

# *Generalized Solutions of non-linear Equations of First Order*

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**1. Introduction.** The main theorems of the classical theory of partial differential equations of first order are valid only locally, in sufficiently restricted domains. The reason for this restriction is that the classical theory requires the solutions to be smooth ones. During the past two decades, however, analysts have obtained global results by relaxing the smoothness conditions on the solutions. Efforts in this direction are motivated by certain problems of fluid dynamics where the physically relevant solution is a generalized one which coincides with the classical solution in its properly restricted domain but which exists, in general, in a larger region. Most of these results pertain to quasilinear equations of first order. The present paper deals with first order equations that are non-linear in the first derivatives.

Denote by  $x, y, \xi, \dots$  points of  $n$ -dimensional real Euclidean space  $E_n$ . For a real-valued function  $z = z(x)$  denote by  $z_x$  its gradient-vector. Existence of  $z_x(x)$  at a point  $x$  is meant in the sense of differentiability of  $z$  at  $x$ , the existence of the differential ( $dz = z_x \cdot dx$ ) at  $x$  as the linear form dominating in  $\Delta z$ . We consider in this paper first order equations

$$(1.1) \quad F(x, z, z_x) = 0,$$

with  $F$  a given real function of its  $2n + 1$  scalar variables that is to be at least continuous in a specified domain of  $(2n + 1)$ -space. Its "solutions" are to include piecewise smooth functions  $z(x)$ . Such functions are continuous throughout. To be specific, suppose that  $z(x)$  is continuous in a full neighborhood of a  $(n - 1)$ -dimensional manifold  $\Phi(x) = 0$  ( $\Phi_x \neq 0$ ) of class  $C'$  and that  $z_x(x)$  is continuous away from  $\Phi = 0$ . Suppose, furthermore, that  $z_x$  has limit values  $z_x^+(x)$  and  $z_x^-(x)$  on the two sides of  $\Phi = 0$ . Then it is a well-known fact that on  $\Phi = 0$  the vectorial jump relation

$$(1.2) \quad [z_x] = z_x^+ - z_x^- = \lambda \Phi_x$$

holds with  $\lambda = \lambda(x)$  a scalar on  $\Phi = 0$ . This is readily proved if a  $C'$ -transformation from  $x$  to new coordinates  $y$  is used in which  $\Phi = 0$  becomes  $y_n = 0$  and

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