

On the Bieberbach Conjecture for Even n

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1. History of the problem. The Bieberbach conjecture asserts that the n th coefficient of any schlicht function

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots + a_nz^n + \cdots$$

defined in the unit circle satisfies the inequality $|a_n| \leq n$, with the equals sign holding only for the Koebe function

$$K(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots$$

and its rotations. Koebe's one-quarter theorem and Bieberbach's proof of the sharp estimate $|a_2| \leq 2$ were the earliest results in this direction [1, 10]. Later Loewner [11] established the deeper inequality $|a_3| \leq 3$, which was also obtained by Schiffer, Schaeffer and Spencer using variational methods [15, 16, 18, 19]. Finally, Garabedian and Schiffer showed that $|a_4| \leq 4$ in a lengthy paper [7], but Charzynski and Schiffer found a much simpler proof [3] of this theorem which indicates that it is actually more elementary than Loewner's result.

The principal contribution of the present paper is to establish that for each positive integer m the conjectured inequality

$$|a_{2m}| \leq 2m$$

is valid provided only that the schlicht function $f(z)$ is close enough to the Koebe function $K(z)$ in some reasonable norm,

$$\|f(z) - K(z)\| < \epsilon_m .$$

Thus for even indices $n = 2m$ the Koebe function yields a strict local maximum of the modulus of the coefficient a_n . We also give an improved version of the Charzynski-Schiffer demonstration that $|a_4| \leq 4$ which suggests how it ought to be generalized to the case of arbitrary coefficients a_{2m} . Moreover, we work the method out in detail for a_6 and show that it reduces the problem to a trigonometric inequality in five independent variables. In the light of extensive computing machine experiments we have made on the latter inequality [14], we argue that such a proof of $|a_6| \leq 6$ is certain to be successful.