

The Algebra of Singular Integral Operators in R^n

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The present note is concerned with investigation of the algebra \mathfrak{A} of singular integral operators acting on $\mathcal{L}^2(\mathbf{R}^n)$ in relation to the Gelfand maximal ideal space of the commutative Banach algebra $\mathfrak{A} | \mathfrak{C}$, where \mathfrak{C} is the closed two-sided ideal of all compact (completely continuous) operators of $\mathfrak{S} = \mathcal{L}^2(\mathbf{R}^n)$. It has been announced previously by Gohberg [7], at least in the case of two dimensions, that this maximal ideal space \mathfrak{M} is in one-to-one correspondence with the cartesian product $S^n \times S^{n-1}$, the Gelfand homomorphism being the mapping κ induced by the Mihlin σ -symbol. Gohberg also states that any $K \in \mathfrak{A}$ is Fredholm if and only if its σ -symbol σ_K does not vanish on \mathfrak{M} , and that the Fredholm index is always zero. This result he obtains also for systems.

Apart from giving an elementary and explicit proof of the above mentioned facts, valid in the case of arbitrary dimension n we also show that \mathfrak{A} is homeomorphic to $S^n \times S^{n-1}$, and that κ is an isometric isomorphism of \mathfrak{A} onto $\mathcal{C}(\mathfrak{M})$.

The isometry can be stated as

$$\inf_{K \in \mathfrak{C}} \|K + C\| = \sup_{S^n \times S^{n-1}} |\sigma_K(x, \xi)|, K \in \mathfrak{A}.$$

It particularly implies that the essential spectrum of a singular integral operator coincides with the image of \mathfrak{M} under the mapping κ in the complex plane.

Two Fredholm operators in \mathfrak{A} are homotopic in \mathfrak{A} if and only if their σ -symbols, considered as mappings from $S^n \times S^{n-1}$ into the multiplicative group of complex numbers are homotopic. Corresponding statements are obtained in case of systems of singular integral operators.

Application of the algebraic machinery developed by M. Breuer and the author in [2b] also yields the expected statements in case of a vector bundle over S^n without any further analytical considerations.

The algebra \mathfrak{A} is obtained by adjoining the Riesz operators to the algebra of all continuous functions on the one-point-compactification of \mathbf{R}^n .

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