

On the Morse Index Theorem*

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We give a generalization of the Morse Index Theorem from one to several independent variables. Before stating our theorem, we give some preliminary definitions.

Let M be a C^∞ compact connected manifold with nonempty boundary and a smooth measure. We say that a domain M' of M with smooth boundary is an open set of M such that the boundary $\partial\bar{M}'$ of the closure of M' is a C^∞ submanifold of M of codimension 1. A deformation of M is a family of domains M_t of M with smooth boundary, $a \leq t \leq b$, such that $\bar{M}_a = M$, and M_t is properly contained in M_s if $t > s$. The deformation is smooth if $\partial\bar{M}_t$ depends in a C^∞ manner on t . The deformation is of ϵ -type if the measure of M_b is less than ϵ .

Let E be a Riemannian vector space bundle over M (so each fiber has assigned an inner product in a smooth way), and E_t be the part of E over \bar{M}_t (if M_t is a deformation). Denote by $C^\infty(E)$ the C^∞ sections of E and by $C_{k-1}^\infty(E)$ those which are zero with the first $(k-1)$ derivatives on ∂M . We define the Morse Index $\beta(L)$ of a strongly elliptic self-adjoint operator $L : C_{k-1}^\infty(E) \rightarrow C^\infty(E)$ (see below) of order $2k$ as follows. Define the bilinear form B_L on $C_{k-1}^\infty(E)$ by

$$B_L(u, v) = \int_M (Lu(x), v(x)) dx.$$

Then $\beta(L)$ is the maximal dimension of a subspace of $C_{k-1}^\infty(E)$ on which B_L is negative definite.

Given M_t , let

$$L_t : C_{k-1}^\infty(E_t) \rightarrow C^\infty(E_t)$$

be the "restriction" of L and let

$$\alpha(t) = \dim \{u \in C_{k-1}^\infty(E_t) \mid L_t u = 0\}.$$

If $\alpha(t) > 0$, t will be called conjugate (relative to M_t of course) with multiplicity $\alpha(t)$. Of course in the Morse Theory L is defined by the Jacobi equation and a conjugate t of M_t will correspond to a geodesic with conjugate end points.

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