

A Dynamic Programming Approach to the Nonparametric Problem in the Calculus of Variations

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1. Introduction. In this paper we present an approach to the so-called nonparametric simple problem of the calculus of variations that is in the spirit of the dynamic programming approach suggested by R. Bellman [1, Chap. IX], [2, Chap. IV]. Our treatment, however, is different from his.

In essence, Bellman assumes that for each point $(t, x) = (t, x_1, \dots, x_n)$ of a region \mathcal{R} in (t, x) -space there exists a solution of the following problem. Let (t_f, x_f) be a fixed point in \mathcal{R} . Let $\chi(t, x)$ denote the class of continuous functions $\xi = \xi(\tau)$ on $[t, t_f]$ such that $\xi(\tau)$ has a piecewise continuous derivative on $[t, t_f]$ and such that the curve defined by $\xi = \xi(\tau)$ lies in \mathcal{R} and joins the point (t, x) to the point (t_f, x_f) . It is required to minimize the integral

$$(1.1) \quad \int_t^{t_f} f(\tau, \xi(\tau), \xi'(\tau)) d\tau,$$

in the class $\chi(t, x)$ where f is a given function and the prime denotes $d/d\tau$.

The value of the integral (1.1) taken along the minimizing curve is a function W of the initial point (t, x) in \mathcal{R} . Bellman assumes that W is C'' in \mathcal{R} , and by formal arguments shows that W must satisfy the following equation

$$(1.2) \quad W_t(t, x) = -\min_{x' \in \Omega} [f(t, x, x') + \sum_{i=1}^n W_{x_i}(t, x)x'_i],$$

where $\Omega = \Omega(t, x)$ is the set of all derivatives $x' = \xi'(t)$ of functions in $\chi(t, x)$.

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