

# *On Markov Chains With the Strong Ratio Limit Property\**

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**Summary.** Some necessary and sufficient conditions are presented for a Markov chain to have the strong ratio limit property.

**1. Introduction.** Throughout this discussion,  $X_n$  will be an irreducible, aperiodic, recurrent Markov chain with integer states and with  $n$ -th step transition matrix  $P^n(x, y)$ . Recall that such a chain is said to have the "strong ratio limit property" if, for arbitrary states  $x, y, z$ , and fixed integer  $m$ ,

$$(1.1) \quad \lim_{n \rightarrow \infty} \frac{P^{n+m}(x, y)}{P^n(z, z)} = \pi_y / \pi_z$$

exists. (We then know that  $\pi_x > 0$  for all  $x$  and

$$\sum_x \pi_x P(x, y) = \pi_y.)$$

The following was established in [5].

**Theorem 1** (Orey). *In order that (1.1) hold for any three states  $x, y, z$ , it is both necessary and sufficient that for some state 0 we have*

$$(1.2) \quad \lim_{n \rightarrow \infty} \frac{P^{n+1}(0, 0)}{P^n(0, 0)} = 1.$$

By the fundamental limit theorem for Markov chains ([1] p. 27, Theorem 1) we know that

$$\lim_{n \rightarrow \infty} P^n(0, 0) = \mu > 0,$$

in every ergodic chain and thus (1.2) clearly holds in every such chain. On the other hand, there are null-recurrent chains in which the ratios in (1.2) fail to

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