

Invariant Subspaces for Normal Operators

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1. Introduction. Let \mathcal{H} be a separable Hilbert space and N a bounded, normal operator on \mathcal{H} ($N^*N = NN^*$). Suppose \mathfrak{M} is a closed subspace of \mathcal{H} invariant under N . Does it follow that \mathfrak{M} is invariant under N^* (*i.e.* that \mathfrak{M}^\perp is invariant under N)? When \mathcal{H} is finite-dimensional, the answer is yes [4, p. 162], and this property characterizes the normal operators on \mathcal{H} . When \mathcal{H} is not finite-dimensional, the simplest example, *e.g.* a bilateral shift, shows that this property need not hold. What is true, in the case of a unitary operator U ($UU^* = U^*U = I$), is that this example is the general case, *i.e.* U is the direct sum of a bilateral shift S of multiplicity m and a unitary V , such that the closed invariant subspaces for V are all invariant also under V^* .

By embedding U into the unitary representation $n \rightarrow U^n$ of the group \mathbb{Z} of integers, we see that the invariant subspaces for U are precisely the “one-sided” invariant subspaces for this representation, *i.e.* those subspaces \mathfrak{M} such that $U^n\mathfrak{M} \subset \mathfrak{M}$ for all $n \geq 0$. One may now ask, given a locally compact abelian group partially ordered by a proper closed semi-group of positive measure, and a unitary representation U of G on \mathcal{H} , when does \mathcal{H} contain proper one-sided invariant subspaces? When U is the regular representation of G on $L^2(G)$, then the subspace \mathfrak{M} of functions vanishing on the positive semi-group is always a proper one-sided invariant subspace (*cf.* [2]). In case $G = \mathbb{R}^1$ or \mathbb{Z} , we prove that up to multiplicity \mathfrak{M} is the only proper one-sided invariant subspace. In particular, it follows that in this case every spectral measure on \hat{G} for a representation U vanishes on a set of positive Haar measure if and only if \mathcal{H} contains no proper one-sided invariant subspaces (this result is a stronger form of theorem 2.1 of [3]).

2. By a representation U of a locally compact group G we shall mean a strongly continuous unitary representation $x \rightarrow U_x$ of G on a separable Hilbert space $\mathcal{H}(U)$. Equivalence of representations means unitary equivalence, quasi-equivalence means equivalence up to multiplicity, and we denote by $\mathfrak{R}(U)$ the

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