

# *The Symbol of the Algebra of Singular Integral Operators\**

E. HERMAN

**Introduction.** The  $\sigma$ -symbol has come to be an important tool in the study of singular integral operators through the pioneering efforts of Mihlin [4] and Calderon and Zygmund [1]. The theory of this symbol has matured to the point where there is now a set of well recognized and easily stated basic properties of the  $\sigma$ -symbol. These properties are listed in Theorem 5 of this paper. The proofs of these properties, unfortunately, have always depended on rather difficult and unilluminating analytic techniques. However, in the  $L^2$  theory, which we study here, a connection has been noticed by more than one author between the  $\sigma$ -symbol and a certain Gel'fand homomorphism. This connection has been exploited quite recently by Gohberg [3], Cordes [2], and Seeley [5] to introduce the high degree of elegance and simplicity that the theory now has. But these authors still rely on difficult analytic techniques for the proof of the fact that  $\|\sigma K\|_0 \leq \|K\|$  for all singular integral operators  $K$ , where  $\sigma K$  is the symbol of  $K$ . In the present paper we show that this property along with the other basic properties of the  $\sigma$ -symbol can be derived by simple algebraic techniques from the well known Gel'fand theory of Banach algebras.

In section 1 we list some of these well known properties of Banach algebras and  $B^*$ -algebras for later reference. In section 2 we derive some simple properties of an axiom system involving abstract Banach algebras. In section 3 we show that singular integral operators can be fitted into the axiom system of section 2 which leads us quickly to our main conclusions in Theorems 4 and 5. In section 4 we briefly consider an abstract setting in which we can deduce Theorem 4.

**1. Basic facts and conventions concerning banach algebras.** In this section  $\mathfrak{A}$  will always denote a (complex) commutative Banach algebra with unit which is not a field. (Apparently, in the standard references on Banach algebras the restriction that  $\mathfrak{A}$  not be a field is omitted. Yet the Gel'fand theory breaks down in this trivial case.) By a  $B^*$ -algebra we will mean a Banach algebra  $\mathfrak{B}$  together with an involution  $*$  :  $\mathfrak{B} \rightarrow \mathfrak{B}$  which satisfies the additional condition that  $\|BB^*\| = \|B\|^2$  for all  $B \in \mathfrak{B}$ , where  $B^*$  denotes  $*(B)$ .

\* The author acknowledges support for this research from the Office of Naval Research under contract NONR 222(64).