

# On the Fourier Representation for Markov Chains and the Strong Ratio Theorem

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**1. Introduction.** For unity of exposition we consider first only discrete parameter denumerable chains; for the continuous parameter analogue see section 7. As usual, the  $n$  step transition probabilities are denoted by  $p_{jk}^{(n)}$ , the first passage probabilities by  $f_{jk}^{(n)}$  (here  $n = 0, 1, \dots$  and  $f_{jk}^{(0)} = 0$ , for all  $j, k$ ). Following Chung [1] we introduce the corresponding quantities with a taboo state  $r$  and denote them by  ${}_r p_{jk}^{(n)}$  and  ${}_r f_{jk}^{(n)}$ ; in terms of the sample paths

$$(1.1) \quad {}_r p_{jk}^{(n)} = P\{X_n = k, X_1 \neq r, \dots, X_{n-1} \neq r \mid X_0 = j\}$$

and similarly for  ${}_r f_{jk}^{(n)}$ . For the generating functions we use capital letters. Thus

$$(1.2) \quad P_{jk}(z) = \sum_{n=0}^{\infty} p_{jk}^{(n)} z^n$$

and so on. (Following Chung we should write  $\hat{p}_{jk}$  instead of  $P_{jk}$ . The change is introduced for typographical convenience since we shall use the complex conjugate  $P_{jk}^*(z)$ .)

From the theory of recurrent events one has the familiar relation

$$(1.3) \quad P_{kk}(z) = \frac{1}{1 - F_{kk}(z)}, \quad |z| < 1.$$

For the sequel it is crucial that there exists the alternative representation

$$(1.4) \quad P_{kk}(z) = \frac{{}_i P_{kk}(z)}{1 - F_{ik}(z)F_{ki}(z)}, \quad |z| < 1.$$

[For an analytic proof see formula (3) in I, §10 of [1]. A direct probabilistic proof is as follows. Consider the recurrent event "the system returns to  $k$  having passed through  $j = k$ ". Its interarrival times have the generating function  $F_{ki}F_{ik}$ . Now the system can be in  $k$  after 0, 1, 2,  $\dots$  such occurrences, whence

$$P_{kk} = {}_i P_{kk} \sum_{n=0}^{\infty} (F_{ki}F_{ik})^n,$$

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