

On an Axially Symmetric Singular Integral Equation

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1. Introduction: The integral equation of first kind,

$$(1.1) \quad \int_0^1 f(t) \log |x - t| dt = g(x), \quad 0 < x < 1,$$

was solved by Carleman [1]. The kernel is related to the fundamental singularity of Laplace's equation which fact suggests that the equation should be of use in boundary-value problems. A simple example is the determination of a two-dimensional harmonic function $u(x, y)$ such that $u(x, 0) = 1$ on $0 < x < 1$. A solution is given by the simple layer potential,

$$(1.2) \quad u(x, y) = \int_0^1 f(t) \log [(x - t)^2 + y^2] dt,$$

where $f(t)$ is a solution of (1.1) for $g(x) = 2$. We observe that the standard integral equation method, in which the solution is expressed as a double layer, fails for this problem. The same boundary-value problem for other equations can be treated in a similar way if the logarithm in (1.2) is replaced by the appropriate fundamental singularity. Then the inversion of (1.1) enables one to convert the resulting integral equation of first kind to a Fredholm equation. Variations of this idea have been used by Fichera [5] in the study of problems for two-dimensional differential equations of higher order.

It would be desirable to have a two-dimensional analog of Carleman's result for use in three-dimensional boundary problems. This would mean a solution of the two-dimensional singular integral equation with the reciprocal distance as kernel. Such an analog is not available. The purpose of this paper is to make a start in this direction by studying the inversion of the operator,

$$(1.3) \quad J_0(\sigma) = (2\pi)^{-1} \int_a^1 \int_0^{2\pi} \sigma(\rho) \rho R^{-1} d\theta d\rho, \quad R^2 = r^2 + \rho^2 - 2r\rho \cos \theta,$$

on $0 \leq a < r < 1$.

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