

# *Square Roots of the Nu and Delta Functions as Generalised Eigenfunctions of Momentum and Position*

D. JUDGE

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**Summary.** An extension of the Hilbert space  $L_2$  is defined in terms of classes of sequences. The extended space contains elements which may be regarded as the square roots of the Dirac delta function and of the Takahashi nu function. These elements furnish eigenfunctions of the position and momentum operators. Expansions expressing an element of  $L_2$  in terms of continuum sums of these eigenfunctions are given.

**1. Introduction.** The mathematical scheme of quantum mechanics centres round a separable Hilbert space of state vectors. Of great importance is the spectral resolution of the Hermitian operators on this space which represent observables. The treatment of the continuous spectrum presents more difficulties than that of the purely discrete spectrum. This problem was approached almost forty years ago in two different ways.

On the one hand, von Neumann [1, 2] gave a rigorous solution in terms of families of projection operators which does not involve any mention of vectors outside the original space  $\mathcal{H}$  ("What does not belong to  $\mathcal{H}$  for us does not exist" [2]), i.e. does not involve any extension of  $\mathcal{H}$ . On the other hand, Dirac [3, 4] gave a formal treatment in which, corresponding to each point in the continuous spectrum, the existence of an "improper eigenfunction" of infinite norm is postulated: thus this solution involves an extension of  $\mathcal{H}$ . Dirac's method, while lacking the rigour of von Neumann's, is more congenial to most physicists; partly because the mathematics is somewhat easier, partly because of the intuitive appeal of the expansion of a state vector as a "sum (or at least an integral) over eigenstates," and partly, indeed, because the improper eigenfunctions, especially of momentum, are of everyday use in applications.

In this paper (a preliminary sketch of the present work was given in D. Judge [5]) an alternative treatment, in the spirit of Dirac's, is developed for a very restricted situation: namely, the case of the position and momentum