

The Homology Groups of A Lattice

JON FOLKMAN

Communicated by G.-C. ROTA

Summary. In [2], Rota defined homology groups for certain subsets, called *cross-cuts*, of a lattice. He showed that the value of the Euler characteristic associated with this homology theory depends only on the lattice, not on the choice of the cross-cut. It was conjectured that the homology groups themselves depend only on the lattice. In this paper we prove Rota's conjecture. Using this result, we determine the structure of the homology groups of an important class of lattices called geometric lattices.

1. Introduction. By a lattice with 0 and 1 we mean a lattice L with elements $0, 1 \in L$ such that $0 \leq x \leq 1$ for all $x \in L$. If L is a lattice with 0 and 1, a *cross-cut* of L is a set $C \subset L$ such that

- (1.1) $0, 1 \notin C$,
- (1.2) If $x, y \in C$ then $x \prec y$ and $y \prec x$,
- (1.3) any finite chain $x_1 < x_2 < \cdots < x_n$ in L can be extended to a chain which contains an element of C .

In [2], Rota defined homology groups $H_n(C)$ for a cross-cut C of a finite lattice with 0 and 1. He showed that the Euler characteristic, $\chi(C) = \text{rank } H_0(C) - \text{rank } H_1(C) + \text{rank } H_2(C) - \cdots$, obtained from this homology theory is an invariant of the lattice. In fact, $\chi(C) = 1 + \mu(0, 1)$, where μ is the Mobius function of the lattice. Rota conjectured that the homology groups themselves are invariants of the lattice, *i.e.*, they do not depend on the choice of the cross-cut C . In this paper we extend the definition of homology groups to cross-cuts of a possibly infinite lattice and prove Rota's conjecture in this more general case.

2. Coverings of abstract complexes. In this section we establish an isomorphism between the homology groups of a complex K and the homology groups of the nerve of a suitable covering of K . This isomorphism is used to prove our main result. Our notation is that of [1].

* This research is sponsored by the United States Air Force under Project RAND—Contract No. AF 49(638) -700.