

# *On the Existence of Optimal Stochastic Controls\**

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**1. Introduction.** Let us begin by recalling a theorem which insures that an optimal control exists for a rather general class of deterministic problems. Using vector notation, let  $X = (X_1, \dots, X_n)$ ,  $U = (U_1, \dots, U_k)$ , and let  $K$  be a compact convex subset of  $k$ -dimensional space  $R^k$  ( $K$  is the "control region"). Consider the system of differential equations

$$(1.0) \quad dX(t) = \alpha(t, X(t)) dU(t), \quad t \geq 0,$$

together with initial data  $X(0) = x_0$ . The vector  $X(t)$  is called the state of the system at time  $t$ , and  $u(t) = U'(t)$  is the control applied at time  $t$ . As admissible controls let us allow all vector-valued functions  $U$  satisfying a Lipschitz condition and the conditions  $U(0) = 0$ ,  $U'(t) \in K$  for almost all  $t$ . The problem is to choose an admissible control  $U$  which minimizes some functional  $\Phi[X(t), U(t), t \geq 0]$ . Assume that the matrix-valued function  $\alpha$  is continuous and satisfies a uniform Lipschitz condition in  $X$ . Then the space of pairs of functions  $X, U$ , with  $U$  admissible, which satisfy (1.0) and the given initial data is compact in the topology of uniform convergence on compact subintervals of  $[0, \infty)$ . Hence, if  $\Phi$  is lower semicontinuous in this topology, then there is a minimizing admissible control  $U_0$  [13]. More generally, the same result still holds if in (1.0)  $\alpha(t, X(t))$  is replaced by some functional  $\alpha[t, X(s), s \leq t]$  of the entire past of  $X$  up to time  $t$ , with the initial data  $X(t) = X^-(t)$  for  $t \leq 0$ . The Lipschitz condition on  $\alpha$  can now be imposed in the form (A.2) of §3 below.

The objective of this paper is to prove an analogous result for stochastic control problems in which (1.0) is replaced by a system of stochastic differential equations

$$(1.1) \quad dX(t) = \alpha(t, X(s), s \leq t) dU(t) + \beta(t, X(s), B(s), s \leq t) dB(t), \quad t \geq 0,$$

where  $B$  is a Brownian motion process and the control  $U$  is a stochastic process belonging to a certain class. The problem is now to minimize the expected value

$$E\Phi[X(t), U(t), 0 \leq t < \infty],$$

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