

Convolutions of Convex Functions*

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1. Introduction. Let K be the class of functions f which are regular and univalent in the disk $E = \{z : |z| < 1\}$ with $f(0) = 0$ and $f(E)$ convex. Using some results of Paatero [3, pp. 67 & 72] we show that if $f \in K$ then f can have at most two discontinuities on the boundary of the disk (the function being defined on the boundary by taking $f(e^{i\theta}) = \lim_{r \rightarrow 1} f(re^{i\theta})$).

Pólya and Schoenberg [4, p. 325] have conjectured that if $\sum_1^\infty a_n z^n \in K$ and $\sum_1^\infty b_n z^n \in K$ then $F(z) = \sum_1^\infty a_n b_n z^n \in K$. Theorem 4 of this paper shows that $F(z)$ given above is close-to-convex. This result is obtained by showing that functions given by a certain integral representation are close-to-convex and then that $F(z)$ has this representation.

Theorem 5 extends the known result: if $f \in K$ and θ is a fixed real number then

$$g(z) = \frac{f(z) - f(e^{i\theta}z)}{1 - e^{i\theta}}$$

is starlike (*i.e.* $g(E)$ is starlike with respect to the origin), to show that for fixed real r and θ , $0 \leq r \leq 1$, the function

$$g(z; r, \theta) = \frac{f(z) - f(re^{i\theta}z)}{1 - re^{i\theta}}$$

is starlike. Since this implies

$$\int_0^z \frac{g(w; r, \theta)}{w} dw \in K,$$

it follows that the conjecture of Pólya and Schoenberg as stated above holds in the case

$$b_n = \frac{1 - r^n e^{in\theta}}{n(1 - re^{i\theta})},$$

where θ is real and $0 \leq r \leq 1$.

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