

An Imbedding of a Stochastic Process $\{X_n\}$ in a Markov Chain

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0. Introduction: Let $\{Z_n\}$ $n = 1, 2, \dots$ be a Markov Chain of the form $Z_n = (X_n, Y_n)$ where X_n and Y_n are real variables. It is in general not true that the "projection" $\{X_n\}$ $n = 1, 2, \dots$ or $\{Y_n\}$ $n = 1, 2, \dots$ is a Markov Chain. So we might ask: What are the conditions on a process $\{X_n\}$ $n = 1, 2, \dots$ in order that there exist a process $\{Y_n\}$ $n = 1, 2, \dots$ such that the two dimensional process $\{Z_n\} = \{(X_n, Y_n)\}$ is a Markov Chain. It turns out that this question is not difficult to answer. The classical way of doing this involves the use of very complicated probability spaces. We shall show that this is unnecessary. In view of this result we might ask: Given two processes $\{X_n\}$ and $\{Y_n\}$, $n = 1, 2, \dots$ what are the conditions on these processes in order that there exist a two-dimensional Markov Chain $\{Z_n\} = \{(U_n, V_n)\}$, $n = 1, 2, \dots$ such that the processes $\{U_n\}$ and $\{V_n\}$ have the same joint distributions as $\{X_n\}$ and $\{Y_n\}$ respectively. We shall seek to answer this question in this paper. We shall show that it is enough to assume that the finite dimensional joint distribution of each of the processes are absolutely continuous and no relationship between the processes need be assumed. However, if the process $\{X_n\}$ and $\{Y_n\}$ have identical distributions we can replace the assumption of absolute continuity of the finite dimensional distributions by a much less restrictive assumption.

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In section 1 we shall answer the first of the above two questions. In section 2 we shall give a brief outline of the motivation for the specific mode of attack that we have chosen. In sections 3 and 4 some preliminaries are established and in section 5 the main result is given.

All distributions considered in this paper are continuous, *i.e.*, have no point masses. We collect a few notations.

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