

# *Structure of $A^*A - AA^*$*

HEYDAR RADJAVI

*Communicated by S. SHERMAN*

1. We propose to give a complete answer (Theorem 8) to the following question: Which bounded (necessarily hermitian) operators on a separable complex Hilbert space  $\mathcal{H}$  are of the form  $A^*A - AA^*$ ? Here  $A$  denotes a bounded operator on  $\mathcal{H}$ . Expressing  $A$  as  $M + iN$ , where  $M$  and  $N$  are hermitian, it is seen that the following question is merely another formulation of the one just given: Which (necessarily skew-hermitian) operators on  $\mathcal{H}$  are commutators of hermitian operators, *i.e.*, are of the form  $AB - BA$ , where  $A$  and  $B$  are hermitian?

It was shown by Wintner [1] that no non-zero scalar multiple of the identity operator is a commutator of hermitian operators. The same result was proved for arbitrary operators by Putnam [2]. A more general result was obtained by Wielandt [3]. Halmos [4] showed that a necessary condition for an operator  $A$  to be a commutator is that for no non-zero scalar  $a$  and compact operator  $C$  it be of the form  $aI + C$ . It was proved recently by Brown and Pearcy [5] that this condition was also sufficient.

The question of commutators of *hermitian* operators, which is certainly distinct from that of commutators of arbitrary operators, is interesting for at least one reason that the magnitude of  $A^*A - AA^*$  can be taken as a measure of the "abnormality" of  $A$ . This question was studied by Putnam, who discovered Corollary 1 to Theorem 1 of the present paper, and also by Halmos to whom Theorem 3 is due [6]. A more general result proven by Putnam was that if  $C = AB - BA$ , where at least one of the two operators  $A$  and  $B$  is semi-normal, then  $\inf \{ |(Cx, x)| : \|x\| = 1 \} = 0$ . (An operator  $A$  is called *semi-normal* if  $A^*A - AA^*$  is either non-negative or non-positive.

While this manuscript was being re-written to make the corrections and alterations suggested by the referee, a paper on commutators of arbitrary operators by Brown, Halmos, and Pearcy [7] has appeared. Although our results were obtained independently, the present paper contains some techniques similar to those employed in [7]; we acknowledge this and reserve the priority credit for [7].