

Unitary Operators Induced by Measure Preserving Transformations

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1. Let (X, \mathbf{M}, m) be a totally finite, separable measure space, and T an invertible, measure preserving transformation on this space. We shall only be concerned with measure preserving transformations modulo sets of measure zero, that is, effectively, with measure preserving automorphisms of the measure algebra of (X, \mathbf{M}, m) . Since X splits into an atomic and a non-atomic part, which are necessarily invariant under T , we can consider separately the cases when X is atomic and when it is non-atomic. If X is atomic, with n atoms say, T is just a permutation, and the set of all invertible, measure preserving transformations of X is a subgroup of the symmetric group S_n ($n \leq \aleph_0$). We therefore assume that X is non-atomic. There is then no loss of generality in assuming that X is the unit interval, \mathbf{M} the class of Lebesgue measurable sets and m Lebesgue measure (*cf.* Halmos [3, p. 171–174], [4, p. 42–44]). Further since every automorphism of the measure algebra of the unit interval is induced by a measure preserving point transformation of the unit interval [10, p. 582–584], we shall speak of “transformations”, when in fact we mean automorphisms.

The invertible transformation (automorphism) T induces a unitary operator U on $L^2(X, \mathbf{M}, m)$ defined by $(Uf)(x) = f(Tx)$ for $f \in L^2(X, \mathbf{M}, m)$. It is known, von Neumann [9, p. 618–619] that every unitary operator on L^2 , satisfying $U(fg) = (Uf)(Ug)$ whenever f, g and fg belong to L^2 , can be induced by a measure preserving transformation, (see §4 for a proof). Some spectral properties of U are also known. It is easy to show that the whole spectrum, as well as its discrete part, are symmetric about the real axis. In fact more is known, the spectrum of the unitary operator of an aperiodic transformation (*i.e.* one which does not satisfy $T^n = I$ for any integer n) is the whole circle. See [4, p. 70–72 and 74–75] from which the result may be easily deduced. For transformations with discrete spectrum there are more detailed results, see [9], [4] and [6]. There are also some results for ergodic and mixing transformations Halmos [4, p. 34 and 39], Jacobs [7, §8.8, §9.3 and §9.8] and Sinai [11], and also some for groups of operators, Foias [2]. However, the general question still remains open as to what unitary