

On the Kinematic Formula in Integral Geometry

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1. Introduction. The kinematic density in euclidean space was first introduced by Poincaré. In modern terminology it is the Haar measure of the group of motions which acts on the space. One of the basic problems in integral geometry is to find explicit formulas for the integrals of geometric quantities over the kinematic density in terms of known integral invariants. An important example is the kinematic formula of Blaschke, as follows [1]: Let E^3 be the euclidean three-space, and let dg be the kinematic density, so normalized that the measure of all positions about a point is $8\pi^2$. (In other words, the measure of all positions of a domain D with volume V such that D contains a fixed point is $8\pi^2 V$.) Let D_i , $i = 1, 2$, be a domain with smooth boundary, of which V_i , $M_1^{(i)}$, $M_2^{(i)}$, χ_i are respectively the volume, the area of the boundary, the integral of mean curvature of the boundary, and the Euler characteristic. Then, if D_1 is fixed and D_2 moves with kinematic density dg , we have

$$(1) \quad \int \chi(D_1 \cap g D_2) dg = 8\pi^2(4\pi\chi_1 V_2 + 4\pi\chi_2 V_1 + M_1^{(1)}M_2^{(2)} + M_2^{(1)}M_1^{(2)}),$$

where $\chi(D_1 \cap g D_2)$ is the Euler characteristic of the intersection $D_1 \cap g D_2$. Formula (1) contains as special or limiting cases many of the formulas in integral geometry in euclidean three-space. It was generalized to euclidean n -space by C. T. Yen and the present author [3] and to the non-euclidean spaces by Santalo [4].

This paper will be concerned with a pair of compact submanifolds (without boundary) M^p , M^q of arbitrary dimensions p , q in euclidean n -space E^n and with the integration with respect to dg of certain geometrical quantities of the submanifold $M^p \cap g(M^q)$. The latter depend only on the induced riemannian metric of $M^p \cap g(M^q)$ and are defined as follows:

Let X be a riemannian manifold of dimension k . In the bundle B of orthonormal frames over X we have the coframes, which consist of k linearly independent linear differential forms φ_α , such that the riemannian metric in X is

$$(2) \quad ds^2 = \sum_{\alpha} \varphi_{\alpha}^2 .$$