

On A Formula of Kac and Achiezer

I. I. HIRSCHMAN, JR.*

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1. Introduction. Let $\mathbf{c}_a(\xi)$ be a complex measurable function on $-\infty < \xi < \infty$ such that

$$\int_{-r}^r |\mathbf{c}_a(\xi)|^2 d\xi < \infty,$$

for every $r > 0$. Let

$$\mathbf{c}(\xi) = \delta(\xi) - \mathbf{c}_a(\xi),$$

where $\delta(\xi)$ is the Dirac delta function and consider the finite section Toeplitz operator with $\mathbf{c}(\xi)$ as kernel which takes $\mathbf{f}(\xi)$ into

$$(1) \quad \mathbf{c} * \mathbf{f}(\xi), \quad 0 \leq \xi \leq r.$$

Here $\mathbf{f}(\xi) = \mathbf{f}_s \delta(\xi) + \mathbf{f}_a(\xi)$ where \mathbf{f}_s is a complex constant and $\mathbf{f}_a \in L^1(0, r)$. It follows from the theory of Fredholm integral equations that (1) is invertible if and only if the Fredholm determinant $D(r) \neq 0$ where

$$D(r) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^r \cdots \int_0^r \det [\mathbf{c}_a(\xi_i - \xi_k)] d\xi_1 \cdots d\xi_n.$$

We now make the assumptions:

- (i) $\int_{-\infty}^{\infty} |\mathbf{c}_a(\xi)| d\xi < \infty, \quad \int_{-\infty}^{\infty} |\mathbf{c}_a(\xi)|^2 (1 + |\xi|) d\xi < \infty,$
- (ii) $c(t) \neq 0, \quad -\infty < t < \infty, \quad [\arg c(t)]_{-\infty}^{\infty} = 0.$

Here

$$c(t) = \int_{-\infty}^{\infty} \mathbf{c}(\xi) e^{i\xi t} d\xi = 1 - \int_{-\infty}^{\infty} \mathbf{c}_a(\xi) e^{i\xi t} d\xi.$$

We further assume the existence of the limits

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