

L_q Estimates for Rearrangements of Trigonometric Series

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1. Main results. A. M. Garsia [1] has recently proved that every L_2 Fourier series can be rearranged so that it converges a.e. The proof rests on an estimate for finite sums $T = \sum_{\mu=1}^n \Phi_{\mu}$, where Φ_1, \dots, Φ_n are orthogonal functions on a measure space. If $\sigma = (\sigma_1, \dots, \sigma_n)$ is a permutation of $\{1, \dots, n\}$, write

$$(1.1) \quad S^*(x, \sigma) = \max_{1 \leq m \leq n} \left| \sum_{\nu=1}^m \Phi_{\sigma_{\nu}}(x) \right|.$$

His result is that

$$(1.2) \quad \frac{1}{n!} \sum_{\sigma} \int [S^*(x, \sigma)]^2 dx \leq A \int |T|^2,$$

for all n and Φ_1, \dots, Φ_n , where the sum is over all such permutations σ , and A is an absolute constant. (From here on we will use $A_r, A(r, s, \dots)$ to denote constants, not necessarily the same at each occurrence, which depend only on the indicated parameters.)

Theorem 1 is an extension of the L_2 estimate (1.2) to L_q for even integers q . First we need a replacement for orthogonality.

(1.3) Definition. Let r be a positive integer. Let (φ_{α}) be a system of complex-valued functions on a measure space. We will say (φ_{α}) is a P -system for r if for any indices $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r$ we have

$$(1.4) \quad \int \varphi_{\alpha_1} \cdots \varphi_{\alpha_r} \bar{\varphi}_{\beta_1} \cdots \bar{\varphi}_{\beta_r} \geq 0.$$

Any orthonormal system is a P -system for 1. The systems

$$(e^{i\nu x}: \nu = 0, \pm 1, \pm 2, \dots) \quad \text{and} \quad (\cos \nu x: \nu = 0, 1, 2, \dots) \quad \text{on} \quad [0, 2\pi],$$

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