

# *Some Theorems about $n$ -Vertex Connected Graphs*

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**1. Introduction.** In this note,  $G$  will denote an undirected graph (finite or infinite) with no loops or multiple edges. Its vertex set will be denoted by  $V(G)$ , or simply by  $V$ . If  $n$  is a positive integer, a connected graph  $G$  is said to be  *$n$ -vertex connected* if  $|V(G)| \geq n + 1$  and if the removal of any set of fewer than  $n$  vertices does not disconnect  $G$ . Such graphs have been investigated by Menger [3] and characterized by H. Whitney [5] as follows:

**Theorem 1.** *If  $|V(G)| \geq n + 1$ , a necessary and sufficient condition that  $G$  be  $n$ -vertex connected is*

- (1-1) *Given  $a, b \in V(G)$ , there exist  $n$  arcs from  $a$  to  $b$  having only  $a$  and  $b$  in common.*

In [1] and [2], G. A. Dirac has proved that some considerably generalized formulations of (1-1) are necessary for  $G$  to be  $n$ -vertex connected. However, not all of these are sufficient. The purpose of this note is thus to generalize (1-1) as a *characterization* of  $n$ -vertex connected graphs. It is further demonstrated that the generalizations obtained are in certain respects the best possible.

**2. Notation and definitions.** As a general reference for the notation and vocabulary of this note, the authors recommend O. Ore [4]. If  $H$  is a subgraph of  $G$ , written  $H \subset G$ , then  $V(H)$  denotes the vertex set of  $H$ . If  $U \subset V(G)$ , then  $G(U)$  denotes the section subgraph of  $G$  with vertex set  $U$ .

To say that  $A = A[a, b]$  is an arc shall mean that  $a$  and  $b$  are its terminal vertices. A family of arcs is said to be *openly disjoint* if the arcs in the family are pairwise disjoint except at common endpoints, if any.

If  $U \subset V(G)$  and  $x \in V(G)$ , we say  $x$  is an *interior vertex* of  $G(U)$  if all the edges incident to  $x$  lie in  $G(U)$ , and  $x$  is an *exterior vertex* of  $G(U)$  if no edge incident to  $x$  lies in  $G(U)$ . Otherwise,  $x$  is an *attachment vertex* of  $G(U)$ .

The following equivalent formulation of an  $n$ -vertex connected graph, due to Ore [4], will be more convenient to use in the sequel.