

*Complex Hilbertian Structures on Stable Linear Dynamical Systems**

J. M. COOK

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The phase space of a linear dynamical system is a real linear space \mathfrak{M} , furnished with a skew-symmetric, non-degenerate, bilinear form B , the Lagrange bracket. Motion of the system is represented by a group \mathcal{G} of symplectic transformations. The purpose of this paper is to give necessary and sufficient conditions that there exist in \mathfrak{M} the structure of a complex Hilbert space such that B is the imaginary part of the inner product and \mathcal{G} is unitary.

A linear transformation S is symplectic [1, 2] if and only if it leaves B invariant: $B(\cdot, \cdot) = B(S(\cdot), S(\cdot))$. When $\dim \mathfrak{M}$ is finite there exists a complete structure theory for symplectic transformations [3, 4, 5, 6, 7]. Unlike the analogous theory for orthogonal transformations, symplectics need not be semi-simple, the λ -matrix may have non-linear elementary divisors. But they cannot occur with a stable symplectic, one such that $\bigcup_{-\infty}^{+\infty} S^n X$ is a bounded set for any fixed X .

The structure of stable symplectics on finite dimensional spaces has been studied by Krein [8], Gelfand & Lidskii [9], and Moser [10] in work of considerable practical importance. In this paper will be given a proof (as outlined in a previous announcement [11]) of the main theorem for the infinite dimensional case: If \mathcal{G} is abelian (or compact, or solvable, or, more generally, "amenable" [12]) and stable, a linear transformation j of \mathfrak{M} can be found to serve as square root of minus one, $j^2 = -1$, such that $\langle \cdot, \cdot \rangle = B(j(\cdot), \cdot) + (-1)^{1/2} B(\cdot, \cdot)$ is hermitian definite and the elements of \mathcal{G} are all unitary with respect to the resulting complex structure. Then the spectral theorem for unitary operators is the desired structure theorem. When \mathcal{G} is time-translation, the dynamical system has been broken up into a direct integral of harmonic oscillators, the simplest of stable systems.

The immediately available natural topology is that defined as the weakest with respect to which B is continuous separately in each of its arguments [13, 14], *i.e.*, the weakest topology in which all linear functionals $B(X, \cdot)$ are continuous

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