On the Fredholm Eigenvalues of Plane Domains

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1. Introduction. Linear integral equations play an important role in potential theory and in conformal mapping. They lead to constructive methods using Neumann series for solving the classical Dirichlet and Neumann problems in the plane and for finding conformal maps of simply connected domains. In many cases the Neumann series converges geometrically, and its rate depends on the least positive non-trivial Fredholm eigenvalue $\lambda_2$ of the corresponding homogeneous integral equation [1, 9, 7]. It is, therefore, of both theoretical and practical importance to estimate $\lambda_2$ and to consider the Fredholm eigenvalues as functionals of the boundary of a domain.

We first present a new lower bound for $\lambda_2$ in a very general setting, depending only on a bounded arc length—chord length ratio of the boundary curve (Theorem 1). We then give criteria concerning “convergence” of boundary curves which imply convergence of eigenvalues (Theorems 2 and 3). As an application we show with a mild condition on the boundary curve that the eigenvalue $\lambda_2$ is the decreasing limit of the corresponding eigenvalues for the level curves of a domain (Theorem 4).

2. The Fredholm eigenvalues. The Fredholm eigenvalues for a domain bounded by a smooth curve are values $\lambda$ which satisfy the equation

\begin{equation}
\varphi(s) = \lambda \int_{0}^{L} \varphi(t) K(s, t) \, dt
\end{equation}

for some continuous (eigen)function $\varphi(s) \not\equiv 0$. In this context the boundary of the domain is assumed to be a (closed) Jordan curve $C$ of class $C^\alpha$ (i.e., $C$ may be represented parametrically in terms of its arc length $s$ by the equation $z = \xi(s), 0 \leq s \leq L$, where $\xi'(s)$ satisfies a Hölder condition of order $\alpha$). Then the Neumann–Poincaré kernel associated with $C$ is

\begin{equation}
K(s, t) = \frac{1}{\pi} \frac{\partial}{\partial n_t} \log |\xi(s) - \xi(t)|,
\end{equation}

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