

On the Loop Spaces of Projective Spaces*

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Communicated by S. S. CHERN

1. Introduction and results. Let \mathcal{K} denote the field \mathcal{R} of real numbers, \mathcal{C} of complex numbers, or \mathcal{Q} of quaternions, let $q = 1, 2$, or 4 be the dimension of \mathcal{K} over the reals, and let $\mathcal{K}P^m$ denote the projective m -space over \mathcal{K} ($m \geq 1$). As is well known (see *e.g.* [2]), there is a fibration

$$(1) \quad S^{q-1} \xrightarrow{i} S^{q(m+1)-1} \xrightarrow{p} \mathcal{K}P^m$$

where S^n is the n -sphere. Since $q - 1 < q(m + 1) - 1$, the inclusion i of the fibre in the total space is nullhomotopic and general results on fibrations (see *e.g.* [4; Theorem 5.2]) imply that the loop space $\Omega\mathcal{K}P^m$ has the homotopy type of the Cartesian product $S^{q-1} \times \Omega S^{q(m+1)-1}$; this remark was first made by G. W. Whitehead [9] in the case $\mathcal{K} = \mathcal{C}$. Two H -spaces X and Y with multiplications denoted by the verticals in the diagram

$$(2) \quad \begin{array}{ccc} X \times X & \xrightarrow{f \times f} & Y \times Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

have the same H -type if there is a base-point preserving homotopy equivalence f yielding homotopy-commutativity in the diagram; since any homotopy inverse $g : Y \rightarrow X$ of f yields homotopy-commutativity in the appropriate diagram, having the same H -type is, indeed, an equivalence relation. Loop spaces are H -spaces under loop multiplication, the spheres S^0, S^1, S^3 are topological groups and, hence, H -spaces, and Cartesian products of H -spaces are H -spaces in the obvious way. We shall prove

Theorem 1.1. *Let \mathcal{K} denote \mathcal{R} or \mathcal{C} and let $q = 1$ or 2 accordingly; then, $\Omega\mathcal{K}P^m$ has the H -type of $S^{q-1} \times \Omega S^{q(m+1)-1}$ if and only if m is odd and ≥ 3 . $\Omega\mathcal{Q}P^m$ has the H -type of $S^3 \times \Omega S^{4m+3}$ if $m \equiv -1 \pmod{24}$.*

Next, let Y be any based topological space and define

$$\text{nil } Y = \sup \text{nil } \pi(X, \Omega Y),$$

where X ranges over all based topological spaces, $\pi(X, \Omega Y)$ is the group of

* This work was partially supported by NSF GP-3902.