

On an Improperly Posed Characteristic Initial Value Problem

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1. It is well known that the Cauchy problem for a linear partial differential equation of the second order of normal hyperbolic type, such as the equation

$$(1.1) \quad \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

is properly posed only if the data are prescribed on a space-like surface, for instance, in the case of (1.1), on the plane $z = 0$. The values of u and of $\partial u / \partial z$ on $z = 0$ then determine (subject to certain conditions of continuity and differentiability) a unique solution of the equation that depends continuously on the data. But analogous Cauchy data on a time-like surface, say the plane $x = 0$, do not constitute a properly posed problem. An improperly posed problem may not have a solution at all. In fact the data cannot be given freely since they have coherence properties, which can be expressed in terms of unique continuation theorems. These are theorems that assert that, if two solutions, u_1 and u_2 , of (1.1) are such that $u_1 = u_2$ and $\partial u_1 / \partial x = \partial u_2 / \partial x$ in a certain domain D of $x = 0$, then necessarily $u_1 = u_2$ and $\partial u_1 / \partial x = \partial u_2 / \partial x$ in a larger domain D^* of $x = 0$. This remarkable result is a particular case of a general theory due to F. John [5].

The object of this note is to show that the argument of John can be applied to another boundary value problem, in which a solution of a hyperbolic equation is to be determined from its values on two intersecting characteristics; this is sometimes called the *characteristic initial value problem*. The existence of a solution of this problem follows, in the analytic case, from a variant of the Cauchy-Kowalewsky theorem (see e.g. [4], p. 116, Theorem 5.1.1). But a distinction between properly and improperly posed problems of this type must again be drawn. For the equation (1.1), this can be done as follows. Let C_1 and C_2 be two smooth characteristics that intersect, and let $P(x_0, y_0, z_0)$ be a point of $C_1 \cap C_2$. A sufficiently small neighborhood U of P is divided by C_1 and C_2 , into four wedge-like domains U_1, U_2, U_3 and U_4 . Now the characteristic cone with vertex P ,

$$(1.2) \quad \Gamma = (z - z_0)^2 - (x - x_0)^2 - (y - y_0)^2,$$