

# Local Spline Approximation by Moments

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By definition, a *spline function of degree  $2m - 1$  on  $[a, b]$*  is a piecewise polynomial function  $s(x) \in C^{2m-2}[a, b]$  which, for some partition

$$(1) \quad \pi : a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

is of polynomial of degree  $2m - 1$  on each interval  $[x_{i-1}, x_i] = I_i$  between successive "joints" of  $\pi$ . An important question of numerical analysis concerns the orders of accuracy of interpolation and approximation to general "smooth" functions by such spline functions. Especially, one looks for error bounds of the form

$$(2) \quad |f(x) - s(x)| \leq M |f^{(2m)}(x)|_{\max} |\pi|^{2m},$$

where  $|\pi| = \max_i |x_i - x_{i-1}|$  as usual, and  $M$  depends only on mesh-length ratios—*e.g.*, is a constant for uniform subdivisions.

It is easy to show that the formula (2) gives the best possible *order* of approximation. Indeed, the best uniform approximation to  $x^{2m}$  by a spline function of degree  $2m - 1$  on  $[x_{i-1}, x_i]$  is given explicitly by the Chebyshev polynomial  $T_{2m}(2(x - m_i)/(x_i - x_{i-1}))$ , where  $m_i = (x_{i-1} + x_i)/2$ . Hence the maximum error is at least [2, p. 365]

$$(3) \quad |f(x) - s(x)|_{\max} \geq |f^{(2m)}(x)|_{\max} |\pi|^{2m} / [2^{4m-1} (2m)!],$$

for  $f(x) = x^{2m}$  on  $[-1, 1]$ —and so (reducing to scale) for  $x^{2m}$  under *any* partition, considering only the deviation on the longest subinterval with  $x_i - x_{i-1} = |\pi|$ .

It seems likely that, at least under mild assumptions on mesh-length ratios, this best possible order of *approximation* is already achieved by spline *interpolation*; at least, this is true for  $m = 1$  and  $m = 2$  [1, Thms. 2, 3]. But the proof even for  $m = 2$  is fairly delicate, and even the case  $m = 3$  of "quintic splines" has resisted attack so far.

The purpose of this note is to describe a new scheme of spline *approximation by moments*, and to show that it does have the "best possible" order of accuracy (2).

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