

Lateral Conditions for Semi-Groups in Partially Ordered Spaces

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Introduction. In the theory of stochastic processes there arise boundary value problems involving integro-differential as well as differential operators, cf. [1]–[4]. The situation is often that a linear, but not necessarily bounded operator A is given on a Banach lattice X , and the question arises as to which restrictions of A are infinitesimal generators of positive contraction semi-groups. More specifically, one wishes to characterize the linear manifolds D such that the restriction A/D is the infinitesimal generator of a semi-group of continuous linear operators $\{T_t : t \geq 0\}$ from X to X such that $\|T_t\| \leq 1$, T_t is strongly continuous for $t \geq 0$, and $f \geq 0$ implies $T_t f \geq 0$. It may happen, of course, that $\{T_t\}$ is a contraction semi-group only on a subspace of X . The strong continuity at $t = 0$ may fail outside the subspace. The most familiar special case is that in which A is a second order ordinary differential operator and $X = C[a, b]$; then the sets D describe the boundary conditions under which the partial differential equation $u_t = Au$ has a unique solution under given initial conditions. Similar problems occur with A an integro-differential operator. It is clearly useful to have a general theorem which would permit a unification and do away with the necessity of treating each operator separately.

The question of characterizing the restrictions of an operator A which generate positive semi-groups is of interest in itself, aside from the many applications. The problem in this general formulation seems rather difficult to attempt; but if we are given a bit more information about A the situation becomes manageable. Let us assume, for example, that we know the existence of one restriction A_0 of A that generates a positive contraction semi-group on a subspace X_0 with a resolvent R_λ^0 defined on all of X such that (a) $\|\lambda R_\lambda^0\| \leq 1$ for $\lambda > 0$, and (b) if $f \geq 0$ then $R_\lambda f \geq 0$ is the unique minimal positive solution of $\lambda u - Au = f$, that is, if u is any other positive solution then $u \geq R_\lambda^0 f$. In this case all the positive semi-groups generated by restrictions of A would have resolvents which dominate R_λ^0 in the sense that $R_\lambda f \geq R_\lambda^0 f$ for each $f \geq 0$. In all the examples referred to above we do indeed have the information (a) and

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