

# *Two-Parameter Singular Perturbation Problems for Second-Order Equations*

R. E. O'MALLEY, JR.\*

*Communicated by* DAVID GILBARG

**Introduction.** Boundary value problems of mathematical physics often depend on a small, positive parameter  $\epsilon$  in such a way that the full differential equation is of higher order than the reduced one obtained by setting the parameter equal to zero. (See, *e.g.*, Friedrichs [5].) In cases where the problem has a limiting solution (as  $\epsilon \rightarrow 0$ ) which solves the reduced equation, regions of non-uniform convergence, or boundary layers (in reference to Prandtl's boundary layer theory), can be expected, in general, near the boundary due to the loss of boundary conditions in the limiting solution. Then, the straightforward, or regular, perturbation procedure must fail and a singular perturbation problem is encountered.

Wasow [11], following the earlier results of Birkhoff [2], Noaillon [7], and Turrittin [9], stated sufficient conditions for convergence to a limiting solution for boundary value problems for homogeneous linear ordinary differential equations depending linearly on a parameter. Latta [6] went further by giving a complete uniformly valid asymptotic expansion for solutions of such equations, and in addition for certain linear partial differential equations. Latta's success, like Prandtl's, was based upon making a change of variables to "blow up" the boundary region in order to accurately describe the nonuniformly converging terms in the solution. Russian mathematicians have been especially active in the area of singular perturbations. Visik and Lyusternik [10], in particular, develop powerful related methods for dealing with these and similar problems.

Specifically, Latta considered the limiting behavior as  $\epsilon \rightarrow 0$  of boundary value problems for equations of the form

$$\epsilon M\Phi + N\Phi = 0$$

with suitable boundary conditions where  $M\Phi$  and  $N\Phi$  are linear differential expressions involving one or more independent variables with  $m$ , the order of  $M$ , greater than  $n$ , that of  $N$ .

---

\*This work was supported in part by the National Science Foundation and the Air Force Office of Scientific Research while the author was at Stanford University.