

New Solutions of the Laplace Equation in Spherical Coordinates

JOHANN MARTINEK*
&
HENRY P. THIELMAN*

Communicated by I. N. SNEDDON

Abstract. New independent solutions $\Psi(r, \theta, \varphi)$ of the Laplace equation in spherical coordinates are found from any given solution $\Phi_0(r, \theta, \varphi)$ in the following way. We construct an operator A such that $\Delta[A\Phi_0(r, \theta, \varphi)]$ is a function of the form $G(r) \cdot F(\theta, \varphi)$; we then let

1. $\Psi(r, \theta, \varphi) = A\Phi_0(r, \theta, \varphi) + g(r) \cdot f(\theta, \varphi)$,

where $f(r)$ and $g(\theta, \varphi)$ are chosen so that

2. $\Delta\Psi(r, \theta, \varphi) = 0$.

This yields $g(r) = r^{-k}$ and inhomogeneous differential equations, which become after proper transformations, equations of the Poisson and Schrödinger type for $f(\theta, \varphi)$. Substituting the solutions of these equations into 1. yields the new independent harmonic functions $\Psi(r, \theta, \varphi)$. These harmonic functions are important for boundary value problems related to the sphere.

1. Consider the Laplace equation for three dimensions in spherical coordinates (r, θ, φ) :

$$(1) \quad \Delta\Phi(r, \theta, \varphi) = \frac{\partial^2\Phi}{\partial r^2} + \frac{2}{r} \frac{\partial\Phi}{\partial r} + \frac{1}{r^2} \left[\frac{\partial^2\Phi}{\partial\theta^2} + \text{ctg } \theta \frac{\partial\Phi}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2\Phi}{\partial\varphi^2} \right] = 0.$$

We define D^* to be a three-dimensional region which excludes the origin but which contains a portion Q of a sphere of radius a with center at the origin, and such that if a point with coordinates (r, θ, φ) is in D^* then the point (R, θ, φ) is also in D^* . (Note that $Rr = a^2$.) Furthermore, D^* is to have the property that every ray from the origin which intersects D^* , intersects it in a single line element of length greater than some constant δ . The portion of D^* which belongs to the open ball $r < a$ will be called D . Furthermore, we let D be an open subset of $r < a$ containing $\bar{D} - \bar{Q}$ (\bar{D} = closure of D , etc.).

* The work reported herein was sponsored by the Air Force Technical Application Center under the Vela Uniform Project of the Advanced Research Projects Agency.