

# Ergodic Properties of Expectation Matrices of a Branching Process with Countably Many Types\*

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**I Introduction.** Let  $I$  be a set of countably many elements and  $(m_{ij})$ ,  $i, j \in I$  be an infinite matrix with non negative entries  $m_{ij}$ . Motivated by applications to branching processes or multiplicative processes we shall call elements of  $I$  "types".  $m_{ij}$  is the expected number of individuals of type  $j$  produced by a single individual of type  $i$ . Any non negative matrix may be the expectation matrix of a branching process. Let  $(m_{ij}^{(n)})$  be the  $n^{\text{th}}$  power of  $(m_{ij})$ . Then  $m_{ij}^{(n)}$  is the expected number of individuals of type  $j$  of the  $(n + 1)^{\text{th}}$  generation if the first generation is a single individual of type  $i$  [10]. The ergodic behavior of  $(m_{ij}^{(n)})$  determines the growth of the process. If  $(m_{ij})$  is a finite matrix there is the theory of positive matrices developed by Perron, Frobenius and Markov. The theory furnishes an asymptotic formula for an irreducible, primitive  $(m_{ij})$ :  $m_{ij}^{(n)} \sim \rho^n v(i)u(j)$  where  $\rho$  is the largest positive eigenvalue and  $v(i)$ ,  $u(i)$  are the corresponding left and right eigenvectors with the normalization  $\sum_i v(i)u(i) = 1$ . Generalizations of the above theory to various abstract settings are given in [1], [2], [3], [12]. In this paper we shall study an *infinite irreducible non negative matrix*. For a function  $f$  defined in  $I$  we define  $Mf$  and  $fM$  by

$$Mf(i) = \sum_j m_{ij}f(j), \quad fM(j) = \sum_i f(i)m_{ij}$$

whenever the right sides of the equalities are well defined. Asymptotic properties of  $\{M^n f\}$  and  $\{fM^n\}$  shall be studied in great detail. Vere-Jones has shown in [16] that there is a unique non negative number  $r$  which is the common radius of convergence of power series  $\sum m_{ij}^{(n)} s^n$ . It will be shown that for all positive numbers  $s \leq r$  there are natural Banach spaces of functions on which  $M$  acts as an operator and for which  $\{M^n f\}$  possesses nice convergence properties. The most interesting ones are for  $s = r$  and the case that  $\sum m_{ij}^{(n)} r^n = \infty$ . There are two strictly positive functions  $u, v$  satisfying  $rMu = u$ ,  $vrM = v$  and  $M$

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