

# *Existence, Uniqueness, Stability, and Monotone Dependence in a Stefan Problem for the Heat Equation\**

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**1. Introduction.** Consider the following Stefan problem for the heat equation: Given the data  $f$ ,  $\varphi$ , and  $b$ , find two functions  $s = s(t)$  and  $u = u(x, t)$  such that the pair  $(s, u)$  satisfies both

$$(1.1) \quad \begin{cases} Lu \equiv u_{xx} - u_t = 0 & \text{in } 0 < x < s(t), \quad 0 < t \leq T \\ u(0, t) = f(t) \geq 0 & \text{and } u(s(t), t) = 0 & \text{for } 0 \leq t \leq T \\ u(x, 0) = \varphi(x) \geq 0 & \text{for } 0 \leq x \leq b, \quad s(0) = b \geq 0 \end{cases}$$

and the free boundary condition

$$(1.2) \quad \dot{s}(t) = -u_x(s(t), t) \quad \text{for } 0 < t \leq T.$$

Here  $T$  is a fixed but arbitrary positive number.

In this paper we prove the global existence and uniqueness of the solution  $(s, u)$  of the problem (1.1), (1.2) and show that the free boundary  $x = s(t)$  depends continuously and monotonically on the data  $f$ ,  $\varphi$ , and  $b$ . The main tool used in our analysis is the maximum principle, both in its strong form [10], and in the form of the parabolic version of Hopf's Lemma [4]. The constructive element in our approach is based on the idea of *retarding the argument* (c.f., equation (3.3)) in the free boundary condition (1.2). Although we present the proofs only in the case of the heat operator in (1.1), our technique applies in principle to the case of a linear parabolic operator of the second order with variable coefficients in which the coefficient of  $u$  is nonpositive so that the maximum principle holds.

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