

# *Deformations of Lie Algebra Structures*

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**0. Introduction.** A Lie algebra is a vector space with an alternating bilinear product which satisfies the Jacobi identity. In this paper we shall consider the set  $\mathfrak{M}$  of all Lie algebra multiplications on a fixed (finite-dimensional) vector space  $V$ . If we choose a basis for  $V$ , then  $\mathfrak{M}$  can be identified with the set  $(c_{ij}^k)$  of all Lie algebra structure constants. The group  $G = GL(V)$  of all vector space automorphisms of  $V$  acts as a transformation group on  $\mathfrak{M}$  and the orbits of  $G$  on  $\mathfrak{M}$  are precisely the isomorphism classes of Lie algebra structures on  $V$ . In the study of "deformations" of Lie algebras, we are interested in geometric properties of the set  $\mathfrak{M}$  considered as a transformation space for  $G$ .

Let  $\mu \in \mathfrak{M}$  and let  $L = (V, \mu)$  be the corresponding Lie algebra. We consider the following problems concerning deformations of  $L$ :

(i) Give sufficient conditions on  $L$  in order that every Lie algebra  $L'$  near  $L$  be isomorphic to  $L$  (we say, in this case, that  $L$  is rigid).

(ii) Describe in terms of algebraic properties of  $L$  the set of all Lie algebras in a neighborhood of  $L$ .

The solutions to both of these problems involve the Lie algebra cohomology space  $H(L, L)$ . Precisely, (i),  $L$  is rigid if  $H^2(L, L) = 0$  and, (ii), a neighborhood of  $L$  (*i.e.*,  $\mu$ ) can be parametrized in the real or complex case by the zeros of an analytic map from  $H^2(L, L)$  to  $H^3(L, L)$ . The computations involved are greatly simplified by the introduction of a graded Lie algebra  $\text{Alt}(V)$  associated with  $V$ .

Strictly speaking, once we establish the relation between Lie algebra structures on  $V$  and the graded Lie algebra  $\text{Alt}(V)$ , all of our results could be obtained by simply quoting the general theorems on deformations in graded Lie algebras developed in [10]. Since so many purely technical details appear there, we have

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