

*Limit Theorems for the Split Times of Branching Processes**

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0. Introduction. Consider a continuous time one dimensional strong Markov branching process $\{X(t); t \geq 0\}$ with the non-negative integers as state space defined on a probability space $(\Omega, \mathfrak{F}, P)$ and standardized to have right continuous sample paths. Let the associated infinitesimal generating function be $u(z) = a[h(z) - z]$ where

$$h(z) = \sum_{i=0}^{\infty} p_i z^i, \quad p_i \geq 0, i = 0, 1, \dots, \sum_{i=0}^{\infty} p_i = 1$$

and $0 < a < \infty$. As usual, we assume henceforth that $h'(1) < \infty$.

It is very suggestive to regard $X(t)$ as the total number of particles at time t in a system where we start with $X(0)$ particles at $t = 0$, each particle lives an exponential length of time with mean a^{-1} and on death creates (or splits into) a random number of new particles whose generating function is $h(z)$, and all particles behave independently of each other and identically. The value $h'(1)$ is then the expected number at each split. A full discussion of definitions and elementary properties of continuous time Markov branching processes is given in [5, Chap. 5], see also [6, Chap. 11].

We will assume henceforth unless stated explicitly to the contrary that $p_0 = 0$ so that extinction of the population is impossible. Because of the Markov property we assume without loss of generality that $p_1 = 0$ as this just amounts to ignoring the case where an individual replicates himself at a split.

It is well known that $\{X(t)e^{-\lambda t}; t \geq 0\}$ ($\lambda = u'(1)$) is a non-negative martingale with respect to the family of σ fields $\mathfrak{F}(t) = \sigma\{X(s, \omega); s \leq t\}$ and hence

$$(1) \quad \lim_{t \rightarrow \infty} X(t, \omega)e^{-\lambda t} = W(\omega) \quad \text{exists with probability one (w.p.1)}$$

and by Fatou's lemma (E denotes the expectation operation)

$$EW(\omega) \leq \lim_{t \rightarrow \infty} E(X(t, \omega)e^{-\lambda t}) = X(0).$$

(Frequently, as customary, we suppress the ω variable.)

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