

The Modulus of a Plane Condenser*

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1. Introduction. This paper is a study of the logarithmic capacity and modulus of a plane condenser. In this study a *condenser* is defined as a triple (R, A, B) , where R is a connected open set of the extended plane \mathbf{P} , whose complement $\sim R$ is the union of nonempty disjoint compacta A and B ; if A and B are connected, the condenser is called a *ring*. If R is contained in the finite plane \mathbf{C} , the *capacity* of the condenser (R, A, B) may be defined as

$$(1) \quad \text{cap } R = \inf_u \iint_R |\nabla u(x)|^2 dx^1 dx^2,$$

where x^i denotes the i th coordinate of the point x (no complex multiplication is used in this paper), and the infimum is taken over all continuously differentiable functions u in R with boundary values 0 at A and 1 at B ; the capacity of an arbitrary condenser may be defined by means of an auxiliary Möbius transformation. The *modulus* of the condenser is

$$\text{mod } R = \frac{2\pi}{\text{cap } R},$$

the constant in the numerator being chosen so that the modulus of the annular ring $\{x : r_1 < |x| < r_2\}$ is equal to $\log(r_2/r_1)$.

Our fundamental theorem is a representation for the modulus of a condenser as an "average" of absolute ratios. If x_1, x_2, x_3, x_4 form an ordered quadruple of points in \mathbf{P} , with $x_2 \neq x_4$, and f is a Möbius transformation such that $f(x_2) = 0$ and $f(x_4) = \infty$, the *absolute ratio* $|x_1, x_2, x_3, x_4|$ is defined by

$$|x_1, x_2, x_3, x_4| = \begin{cases} 1 & \text{if } f(x_1) = f(x_3), \\ |f(x_1)|/|f(x_3)| & \text{if } f(x_1) \neq f(x_3). \end{cases}$$

Then the fundamental theorem is given by the formula

$$(2) \quad \text{mod } R = \inf_{\sigma} \int_B \int_B \int_A \int_A \log |a_1, b_2, b_1, a_2| d\sigma(a_1) d\sigma(a_2) d\sigma(b_1) d\sigma(b_2),$$

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