

Remarks on a Stefan Problem

J. R. CANNON & C. DENSON HILL*

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1. Introduction. This paper is in two parts and is concerned with Stefan problems of the form

$$(1.1) \quad \begin{cases} Lu \equiv u_{xx} - u_t = 0 & \text{in } 0 < x < s(t) \text{ for } 0 < t < \infty \\ u(0, t) = f(t) \geq 0 \text{ and } u(s(t), t) = 0 & \text{for } 0 < t < \infty \\ u(x, 0) = \phi(x) \geq 0 \text{ for } 0 \leq x \leq b, \quad s(0) = b \geq 0 \end{cases}$$

$$(1.2) \quad \dot{s}(t) = -u_x(s(t), t) \text{ for } 0 < t < \infty.$$

In Part II we obtain precise results on the asymptotic behavior of the free boundary $x = s(t)$ as $t \rightarrow \infty$. Theorem 4 gives a necessary and sufficient condition on $f(t)$ in order that $s(t)$ approach a finite limit. Theorem 6 describes the asymptotic behavior of $s(t)$ for any $f(t)$ which is bounded and goes to zero as $t \rightarrow \infty$. Theorem 7 shows, in particular, that $s(t)$ is asymptotic to a parabola whenever $f(t)$ is asymptotic to a positive constant.

When $b = 0$ the above Stefan problem contains a difficulty not present if $b > 0$. This is due to the fact that when $b = 0$ the free boundary $x = s(t)$ actually emanates from a point at which $u(x, t)$ may be discontinuous (unless $f(t)$ has a zero at $t = 0$).

In Part I we prove an existence theorem for the case where $b = 0$. That existence theorem is actually used in Part II in treating the asymptotic behavior of $s(t)$. In Part I we also give a self-contained proof of the monotone dependence of the free boundary on the data which uses nothing more than the parabolic version of Hopf's lemma [4]. From the monotone dependence follows an especially elegant proof of uniqueness that covers the cases $b = 0$ and $b > 0$ simultaneously.

The results of Part I and Part II settle some problems posed as open questions by Friedman in [2] and [3].

Although existence and uniqueness in the case $b > 0$ has been widely discussed (for references, see [4]), the case $b = 0$ does not seem to have been thoroughly treated. In [1] the case $b = 0$ was treated under the hypothesis that, at $t = 0$, $f(t)$ had a zero of a prescribed order. However, Stefan himself, in his original

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