

# *A Limitation of Fourier Analysis*

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**1. Statement of conjecture.** In his classic paper [7] on generalized harmonic analysis, Wiener extended the notion of *energy spectrum* to a wide class of *random functions*. Typical physical examples of these are the outputs from filters having “white noise” (Brownian motion) inputs. Wiener defined the *autocorrelation function* of a sample function  $f(x) = f(x, \omega)$  as the function

$$(1) \quad R(h, f) = R^*(-h, f) = \lim_{X \rightarrow \infty} \frac{1}{2X} \int_{-X}^X f(x)f^*(x+h) dx.$$

If this function exists and is in  $L_1 \cap L_2$ , then the Fourier transform

$$(2) \quad \hat{R}(q, f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iqh} R(h, f) dh,$$

is nonnegative by a well-known result of Bochner. This fact and consideration of the sample function  $f(x) = ce^{iax}$  led Wiener to suggest that  $\hat{R}(q, f)$  should be interpreted as the mean squared amplitude of the Fourier component with wave-number  $q$ . That is, one should think of  $\hat{R}(q, f)$  as the mean *power density* of  $f$  (per unit wave-number) at the wave-number  $q$ .

Wiener justified these ideas rigorously by showing, for a given continuous power density  $E(q)$ , how to construct *normal homogeneous random functions* (N.H.R.F.)  $f(x, \omega)$  and an associated probability measure  $\mu_E$  on an abstract sample space, such that  $\hat{R}(q, f)$  as defined by (1)-(2) exists and equals  $E(q)$  with  $\mu_E$ -probability one. In physical language, “time averages of sample functions are ensemble averages with probability one”: metric transitivity.

Wiener went further: he showed how to write almost every sample function  $f(x, \omega)$  as a symbolic Fourier-Stieltjes transform

$$(3) \quad f(x, \omega) = \int_{-\infty}^{\infty} e^{iax} dF(q, \omega).$$

Namely, he defined  $F(q, \omega)$  up to an additive constant for almost every sample function by the formula

$$(3') \quad F(q'', \omega) - F(q', \omega) = \frac{i}{2\pi} \text{l.i.m.}_{X \rightarrow \infty} \int_{-X}^X f(x, \omega) [e^{-ia''x} - e^{-ia'x}] \frac{dx}{x}.$$

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