

The Three-Dimensional Wave Equation in a Characteristic Quarter-Space

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Introduction and Summary. In this paper we study the wave equation,

$$\square u \equiv u_{tt} - u_{xx} - \sum_{i=1}^2 u_{y_i y_i} = 0,$$

in a characteristic quadrant of space-time, $|x| \leq |t|$. This problem, so well known in the one-dimensional case, has not been studied much in more than one spatial dimension.

In one dimension, there is no difference between t and x in the equation $u_{tt} = u_{xx}$. Using the method of characteristics one easily constructs a unique solution u for arbitrary given boundary values, either in the quadrant $|x| < t$ or in $|t| < x$.

In higher dimensions, these two problems become essentially different. The "interior problem," $|x| < t$, is still solvable for general boundary values. This is perhaps well known, but we include a simple proof for the sake of completeness. (See Hadamard, [2], para. 119, for a proof using finite parts of divergent integrals; see Riesz, [6], for a proof using Riemann–Liouville operators and analytic continuation.)

The "exterior problem," $|t| < x$, is no longer well posed in higher dimensions. We show that a solution exists for analytic boundary values, but not for C^∞ boundary values with compact support. (Pucci [5] gives similar results by a different method. For further references on problems in exterior domains, see Lax and Phillips, [4]. Friedlander [7] has an interesting example of a physical problem leading to an incorrectly posed Goursat problem.)

Having proved that data cannot be prescribed arbitrarily on the whole boundary, we then show that the boundary values on half the boundary ($t = x$, say) already determine the solution uniquely—but only if in addition some restriction is placed on the growth of u and u_t at ∞ . This leads to the conclusion (Corollary 2) that if u exists in all space-time, it is uniquely determined by the data needed to determine it in one interior quadrant!

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