

On Trigonometric Series Associated with Weak Closed Subspaces of Continuous Functions*

HASKELL P. ROSENTHAL*

Communicated by HANS LEWY

If f is a complex Lebesgue integrable function on the unit circle, $\hat{f}(n)$, the n^{th} Fourier coefficient of f , is defined by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta \quad (n = 0, \pm 1, \pm 2, \dots).$$

Let C be the Banach space of continuous functions on the unit circle, A those functions in C with absolutely convergent Fourier series, i.e., those f in C for which $\sum_{-\infty}^{\infty} |\hat{f}(n)| < \infty$, and L^{∞} (resp. L^1) the Banach spaces of equivalence classes of bounded measurable (resp. integrable) functions on the circle; L^{∞} is endowed with the essential supremum norm, denoted $\|\cdot\|_{\infty}$; and we regard C as being a closed subspace of L^{∞} .

If \mathfrak{B} is one of the spaces, C , A , L^{∞} or L^1 , and E is a subset of \mathbf{Z} , the integers, we define

$$\mathfrak{B}_E = \{f \in \mathfrak{B} : \hat{f}(n) = 0 \text{ for all } n \notin E\}.$$

We show here that there exists a set E such that $L^{\infty}_E \subset C_E$, but $C_E \not\subset A$. Indeed, examples of such sets may be given explicitly as follows:

Given S a subset of \mathbf{Z} , and $l \in \mathbf{Z}$, define $lS = \{ln : n \in S\}$. Let $E_n = \{1, 2, \dots, n\}$. Then $E^1 = \bigcup_{n=0}^{\infty} (19)^n n! E_{n+1}$ and $E^2 = \bigcup_{n=1}^{\infty} (2n)! E_{2n}$ both have this property.

In Theorem 3, we give a general class of sets E for which $L^{\infty}_E \subset C_E$; the above examples follow as a corollary to this result.

We wish to thank Professor Y. Katznelson for a stimulating discussion on a related topic; indeed, the techniques we use are similar to those shown us by him, concerning a different problem.

* This research was supported by National Science Foundation grant GP-5207.