

Collectively Compact and Totally Bounded Sets of Linear Operators

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1. Introduction. Let X and Y be real or complex normed linear spaces and $[X, Y]$ the space of bounded linear operators on X into Y with the operator norm topology. A set $\mathcal{K} \subset [X, Y]$ is *collectively compact* iff the set $\{Kx : K \in \mathcal{K}, \|x\| \leq 1\}$ has compact closure (hence, is totally bounded).

Collectively compact sets of operators were introduced in [1] in connection with the approximate solution of integral equations. Their general properties are studied in [2], which also contains an up-to-date bibliography on the subject. A related paper [3] concerns operators T and T_n , $n = 1, 2, \dots$, such that $T_n \rightarrow T$ strongly and $\{T_n - T : n \geq 1\}$ is collectively compact. It was shown that such operators have many of the properties of those for which $\|T_n - T\| \rightarrow 0$.

Since the analysis simplifies in the latter case, it is important to determine when $T_n \rightarrow T$ strongly but $\|T_n - T\| \not\rightarrow 0$. It is easy to prove that $\|T_n - T\| \rightarrow 0$ iff $T_n \rightarrow T$ strongly and $\{T_n - T : n \geq 1\}$ is totally bounded in the topology of $[X, Y]$. Thus, we are led to compare collectively compact and totally bounded sets in $[X, Y]$.

The following results were obtained in [2]. Every compact set of compact operators in $[X, Y]$ is collectively compact. For Y complete, every totally bounded set of compact operators in $[X, Y]$ is collectively compact. For $Y = X$, a Hilbert space, a set \mathcal{K} of compact operators in $[X, X]$ is totally bounded iff both \mathcal{K} and $\mathcal{K}^* = \{K^* : K \in \mathcal{K}\}$ are collectively compact.

Conjecture. *For arbitrary normed linear spaces X and Y , if $\mathcal{K} \subset [X, Y]$ and $\mathcal{K}^* \subset [Y^*, X^*]$ are collectively compact, then \mathcal{K} is totally bounded.*

(Since $\|K^*\| = \|K\|$, \mathcal{K} is totally bounded iff \mathcal{K}^* is totally bounded.)

The conjecture is proved in this paper for sets of operators such that $\dim KX \leq n$ for all $K \in \mathcal{K}$ ($n = 1, 2, \dots$) and for certain limits of such sets. A related paper [4] will use these results together with spectral theory to establish the conjecture in more general cases.

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