

# *A Geometric Criterion for Algebraic Varieties*<sup>1</sup>

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*Communicated by A. H. WALLACE*

## I. INTRODUCTION

The origin of the criterion in question is the following fact which I observed after completing [7] and which can be proved by the method used in [7]:

**Theorem 1.** *If  $f$  is an entire function of  $n$  complex variables and if there is a constant  $R < \infty$  such that every point of the zero-set of  $f$  has at least one coordinate whose absolute value is less than  $R$ , then  $f$  is the product of a polynomial and a zero-free entire function. In particular, the zero-set of  $f$  is an algebraic variety.*

The generalization of Theorem 1 which is the main result of this paper (Theorem 2) characterizes pure-dimensional *algebraic* varieties in  $C^n$  (the space of  $n$  complex variables) as precisely those pure-dimensional *analytic* varieties in  $C^n$  which lie in regions defined by certain inequalities involving only the absolute values of the coordinates.

Before stating this and other results, it seems advisable to recall some definitions and to establish some terminology.

All functions considered in this paper will be complex-valued.

The zero-set of a function  $f$ , often denoted by  $Z(f)$ , is the set of all points  $p$  in the domain of  $f$  at which  $f(p) = 0$ .

If  $\Omega$  is an open set in  $C^n$ , the class of all holomorphic functions with domain  $\Omega$  is denoted by  $H(\Omega)$ . In particular,  $H(C^n)$  is the class of all *entire functions* of  $n$  variables. If  $P$  is a polynomial in the variables  $z_1, \dots, z_n$ , with complex coefficients, we shall sometimes simply say that  $P$  is a polynomial in  $C^n$ .

An *analytic variety* in  $C^n$  is a *closed* set  $V \subset C^n$  with the following property: to every  $z \in V$  correspond functions  $f_1, \dots, f_r$ , defined and holomorphic in some neighborhood  $\omega$  of  $z$  in  $C^n$ , such that

$$(1) \quad V \cap \omega = Z(f_1) \cap \dots \cap Z(f_r).$$

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<sup>1</sup> Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant No. 1160-66, and by the Wisconsin Alumni Research Foundation.