

On Local Spline Approximation by Moments

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1. This note is intended to generalize the statements of [1]. Incidentally it should justify some of the steps taken in [1].

2. Let m be a positive integer, $\pi : 0 = x_0 < x_1 < \dots < x_n = 1$ a partition of the unit interval, and denote by $S = S_\pi$ the set of spline functions on $[0, 1]$ of degree $2m - 1$ with (interior) joints x_1, \dots, x_{n-1} . We wish to investigate the behavior of

$$(1) \quad \text{dist}(f, S) = \min_{s \in S} \|f - s\|_\infty,$$

for $f \in C[0, 1]$, as the mesh of π , $|\pi| = \max_i |x_{i+1} - x_i|$, goes to zero. As is pointed out in [1],

$$(2) \quad \text{dist}(f, S_\pi) = O(|\pi|^k)$$

will not hold for $k > 2m$, except for the trivial case that f is a polynomial of degree $\leq 2m - 1$. It is further stated there that if $f \in C^{2m}[0, 1]$ and if the numbers

$$(3) \quad M_\pi = \max_{|i-j|=1} (x_{i+1} - x_i)/(x_{j+1} - x_j)$$

stay bounded, then there exists K independent of f or π and $s_\pi \in S_\pi$ s.t.

$$|f(x) - s_\pi(x)| \leq K |\pi|^{2m} \|f^{(2m)}\|_\infty, \quad \text{all } x \in [x_m, x_{n-m}].$$

It is one result of this note that in fact

$$(4) \quad \text{dist}(f, S_\pi) = O(|\pi|^{2m}),$$

for $f \in C^{2m}[0, 1]$, and that (4) holds even without the assumption of bounded mesh ratios M_π .

The argument in [1] relies on a linear approximation scheme, called local spline approximation by moments, which realizes the convergence rate $O(|\pi|^{2m})$. Briefly, the approximation $P_\pi f$ to f is defined by

$$(5) \quad (P_\pi f)(x) = p(x) + \sum_i G(x, x_i) \int_0^1 W_i(t) f^{(2m)}(t) dt.$$

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