

Unitary Operators Induced by Measurable Transformations

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1. Let (X, \mathbf{M}, m) be a totally finite, separable measure space, and T an invertible measurable non-singular transformation on this space (i.e. one for which $m(E) = 0$ if and only if $m(T^{-1}E) = 0$). We shall only be concerned with such transformations modulo sets of measure zero, that is, effectively, with automorphisms, not necessarily measure preserving, of the measure algebra of (X, \mathbf{M}, m) . As was remarked in the introduction to an earlier paper [3], there is no loss of generality in assuming firstly that the space X is non-atomic, and further (in view of a well-known isomorphism theorem [8] p. 171–174, [10] p. 42–44) that X is the unit interval, \mathbf{M} the class of Lebesgue measurable sets and m Lebesgue measure. Since every automorphism of the measure algebra of the unit interval is induced by a measurable non-singular point transformation of the interval, we shall speak of “transformations” when in fact we mean automorphisms.

Since $m(T^{-1}E)$ is equivalent to $m(E)$, it follows by the Radon–Nikodym theorem that there exists a function $\omega_T(x)$ such that for all $E \in \mathbf{M}$, $m(T^{-1}E) = \int_E \omega_T(x) \, dm(x)$; $\omega_T(x)$ is called the *Radon–Nikodym kernel of T with respect to m* . The properties of ω_T are well known, see e.g. Dowker [6]. We need particularly the following. ω_T is strictly positive almost everywhere and $\omega_T \in L^1([0, 1], \mathbf{M}, m)$, so $\omega_T^{1/2} \in L^2([0, 1], \mathbf{M}, m)$ (for any non-negative function f , we write $f^{1/2}$ for its positive square root). Further if $f \in L^1$ then so does $f(T^{-1}x)\omega_T(x)$ and $\int f(T^{-1}x)\omega_T(x) \, dm(x) = \int f(x) \, dm(x)$, hence if $f \in L^2$ so does $f(T^{-1}x)\omega_T^{1/2}(x)$ and the respective L^2 norms are equal. The operator U defined in $L^2([0, 1], \mathbf{M}, m)$ by $(Uf)(x) = f(T^{-1}x)\omega_T^{1/2}(x)$ is an invertible isometry of L^2 and so is unitary. It is called the unitary operator induced by T . If $h(x)$ is any function in $L^2([0, 1], \mathbf{M}, m)$ such that $|h(x)|^2 = \omega_T(x)$ then the operator U_h , given by $(U_h f)(x) = f(T^{-1}x)h(x)$ is also unitary; it is called a unitary operator associated with T . [Note. The above definitions are those used by most authors today. In earlier work it was customary to define ω_T by $m(TE) = \int_E \omega_T(x) \, dm(x)$

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