

# *Spectral Properties of Collectively Compact Sets of Linear Operators\**

P. M. ANSELONE & T. W. PALMER

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**1. Introduction.** Let  $X$  be a real or complex normed linear space,  $[X]$  the space of bounded linear operators from  $X$  into  $X$ , and  $\mathfrak{B} = \{x \in X : \|x\| \leq 1\}$ . A set  $\mathfrak{K} \subseteq [X]$  is collectively compact iff the set

$$\mathfrak{K}\mathfrak{B} = \{Kx : K \in \mathfrak{K}, x \in \mathfrak{B}\}$$

has compact closure.

Collectively compact sets of operators have been studied in connection with the approximate solution of integral equations [2, Bibliography]. A detailed theory, which covers several such applications, relates the spectral properties of an operator  $T$  to those of operators  $T_n$  in a sequence converging strongly to  $T$  and having  $\{T_n - T : n = 1, 2, \dots\}$  collectively compact [3].

The relationship between collective compactness and total boundedness is of particular interest since a sequence  $\{T_n\}$ , satisfying the hypotheses of [3] stated above, converges to  $T$  in norm iff the set of differences  $\{T_n - T\}$  is totally bounded. If  $X$  is complete it is shown in [2] that any totally bounded set of compact operators is collectively compact. As a partial converse it is also shown that a collectively compact set of normal operators in Hilbert space is totally bounded. From this it follows that a collectively compact set  $\mathfrak{K}$  of operators in a complex Hilbert space is totally bounded iff the set  $\mathfrak{K}^* = \{K^* : K \in \mathfrak{K}\}$  is collectively compact. This last result has been generalized to the case of an arbitrary Banach space in [1] with the restriction that  $\{\dim KX : K \in \mathfrak{K}\}$  should be bounded. In [7] this restriction is removed.

After the introduction of necessary notation in Section 2, it is shown in Section 3 that several spectral properties of individual compact operators hold uniformly for all operators in a collectively compact set. These results are used in Section 4 to prove that a set of normal operators on a complex Banach space with a uniformly smooth unit ball is a totally bounded set of compact operators iff it is collectively compact.

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