

# *Elliptic-Parabolic Equations of the Second Order*

R. S. PHILLIPS & LEONARD SARASON<sup>1</sup>

The purpose of this paper is to study the uniqueness of the weak solution of the Dirichlet boundary-value problem for the second order elliptic-parabolic partial differential equation

$$(1) \quad Lu = \alpha^{ij}u_{,i} + \beta_i u_{,i} + \gamma u = f, \quad \alpha^{ij}\xi_i \xi_j \geq 0,$$

in a bounded domain  $G$  in  $R^m$  with smooth boundary  $\Sigma$ ; here subscripts denote differentiation and the usual summation convention is employed.

The investigation into this class of problems was initiated by Fichera [1, 2] in 1956 who established the existence of weak solutions and in the process obtained certain estimates on the solutions. More recently Oleinik [6, 8] proved for a large class of problems that the weak solution of (1) is unique. In the present paper we consider a slightly different class of weak solutions and as a consequence are able to establish the uniqueness for a larger class of problems than that treated by Oleinik. For example we do not require, as does Oleinik, that the  $\alpha^{ij}$  be smoothly continuable to a larger domain (including  $\bar{G}$ ) in which  $\alpha^{ij}\xi_i \xi_j \geq 0$ . However we require of the weak solution (*cf.* Fichera [2]) that both  $u^2$  and  $\alpha^{ij}u_{,i}u_{,j}$  be integrable in  $G$  whereas the comparable requirement of Oleinik's is that  $u^3$  be integrable in  $G$ .

The smoothness of solutions to (1) has been studied by Oleinik [7, 8] and Kohn and Nirenberg [4] for a restricted class of problems of this type. We shall avail ourselves of results from these papers as well as the machinery of first order symmetric systems of partial differential equations developed by Friedrichs [3], Lax and Phillips [5], Sarason [10], and Phillips and Sarason [9].

We shall need the following notation: Let  $n = (n_1, n_2, \dots, n_m)$  be the outward normal to  $\Sigma$  and set  $\beta = (\beta^i - \alpha^{ij}n_j)n_i$ . It is customary (*cf.* Fichera [1, 2]) to let  $\Sigma_3$  denote the non-characteristic boundary points, that is those points in  $\Sigma$  at which  $\alpha^{ij}n_j n_i \neq 0$  and to denote the set of characteristic points of  $\Sigma$  at which  $\beta = 0$ ,  $\beta < 0$  and  $\beta > 0$  by  $\Sigma_0$ ,  $\Sigma_1$  and  $\Sigma_2$ , respectively; these sets are invariant under coordinate transformations in  $C^2$ . We denote the union

---

<sup>1</sup> Sponsored by the National Science Foundation (GP-6151 and GP-5841) and the Office of Naval Research (Nonr-225-79).