

A New Approach to Local Times

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Communicated by WILLIAM FELLER

Introduction. A real valued strong Markov process $(X(t))$ is said to admit local times relative to a measure m if for almost every trajectory $X(\cdot)$ there is a function $J(t, x)$, jointly continuous in the time variable t and the space variable x , and such that

$$\int_0^t \varphi[X(u)] du = \int J(t, y)\varphi(y)m(dy)$$

for each $t > 0$ and each bounded measurable function φ . We call $J(t, x)$ the "local time at x up to time t ."

Much work has been done on the problem of deciding when local times exist and of determining their properties. In this paper we introduce a new technique for attacking this problem. To introduce this technique we assume for the moment that the local times $J(t, x)$ are known to exist. We define $\tau_x(s)$, the "instant that local time at x passes s ," by

$$(0.1) \quad \tau_x(s) = \sup \{t : J(t, x) \leq s\}$$

and $T_{xy}(s)$, the "local time at y when local time at x passes s ," by

$$(0.2) \quad T_{xy}(s) = J(\tau_x(s), y).$$

It is easy to see that the individual $\tau_x(s)$ are Markov times, that $T_{xy}(s)$ depends only on the past up to time $\tau_x(s)$, and that $T_{xy}(s+r) - T_{xy}(s)$ depends only on the future after $\tau_x(s)$. Also if the process $\{X(t)\}$ satisfies reasonable regularity conditions, then $X[\tau_x(s)] = x$ almost everywhere. From the strong Markov property it follows that the stochastic process $\{T_{xy}(s), s \geq 0\}$ has stationary independent increments. From the well known results of P. Lévy for such processes (see for example Ito and McKean [8, p. 31]), it follows that there is a constant $a \geq 0$ and a measure $\pi(dl)$, concentrated on the open half line $(0, \infty)$ and satisfying $\int \pi(dl)l/(1+l) < \infty$, such that the Laplace transform of these

* Research associated with a project in probability at Princeton University supported by the office of Army Research.