

*On the Differentiability and Compactness of Semi-Groups of Linear Operators**

A. PAZY

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1. Introduction. Let X be a Banach space, $T(t)$, $0 \leq t < \infty$ a strongly continuous semigroup of bounded linear operators on X . Let A be the infinitesimal generator of $T(t)$. A is usually a closed linear unbounded operator with domain $D[A]$ dense in X . Similarly for $t > 0$ the operator $T'(t) = AT(t)$ is ordinarily an unbounded linear operator whose domain includes $D[A]$.

In this note we shall deal mainly with two questions. The first question is to characterize the generator of a semi-group $T(t)$ for which $T'(t)$ is a bounded operator defined on all of X for $t \geq t_0 > 0$ (differentiability) or for every $t > 0$ (C^∞). In Section 2 we give necessary and sufficient conditions for differentiability and C^∞ .

The second question is to give necessary and sufficient conditions for $T(t)$ to be compact for every $t \geq t_0 > 0$ or for every $t > 0$. In Section 3 we deal with this question.

In Section 4 we are concerned with the stability of the previous properties with respect to perturbations of the infinitesimal generator A by a bounded operator B . We show that differentiability and ultimate compactness are unstable while compactness for every $t > 0$ is a stable property.

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2. Differentiability. In the remainder of this paper we consider strongly continuous semigroups, $T(t)$, of bounded operators on X (see [1], Chapter VIII) satisfying

$$(2.1) \quad \|T(t)\| \leq Me^{\omega t}.$$

The infinitesimal generator A of $T(t)$ is defined by

$$(2.2) \quad Ax = \lim_{t \rightarrow 0^+} t^{-1}(T(t) - I)x,$$

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