

Quasi-Linear Elliptic Boundary Value Problems

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1. Introduction. Let Ω be a region in R^n and $\partial\Omega$ denote the boundary of Ω . We consider quasi-linear elliptic boundary value problems of the form

$$(1) \quad L[u] \equiv \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta u) = f(x, u, \dots),$$

$$x \in \Omega, \quad a_{\alpha\beta}(x) = a_{\beta\alpha}(x), \quad |\alpha|, |\beta| \leq m,$$

$$(2) \quad L[u] \equiv \lambda f(x, u, \dots), \quad x \in \Omega,$$

subject to the boundary conditions

$$(3) \quad D^j u(x) = 0, \quad x \in \partial\Omega, \quad 0 \leq j \leq m-1,$$

where we have freely used multi-index notation, cf. [1], [2], or [3], and $f(x, u, \dots)$ denotes a function of x, u , and possibly all derivatives $D^\alpha u$ with $|\alpha| \leq m$.

This class of problems has been studied in [1], [2], and [3] under the restrictions that the coefficients $a_{\alpha\beta}(x)$ are measurable and uniformly bounded in Ω , that there exists a positive constant C such that

$$(4) \quad (L[w], w)_{L^2(\Omega)} \geq C \|w\|_{W^{m,2}(\Omega)}^2 \equiv C \left(\sum_{|\alpha| \leq m} \int_{\Omega} |D^\alpha w(x)|^2 dx \right)$$

for all $w \in W_0^{m,2}(\Omega)$, i.e., for all w in the closure of $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{W^{m,2}(\Omega)}$, and that f depends on x and $D^\alpha u$ with $|\alpha| \leq m-1$, but *not* $D^\alpha u$ with $|\alpha| = m$.

In this paper, we extend the results of [1] to problems in which the differential operator L satisfies a weaker "positive definite" hypothesis than (4) and f depends on x and $D^\alpha u$ with $|\alpha| \leq m$. The price of this extension is a slightly stronger hypothesis on the smoothness of the coefficients $a_{\alpha\beta}(x)$.

2. Main results. Throughout this paper the coefficients $a_{\alpha\beta}(x)$ are assumed (I) to be bounded, measurable functions such that the domain of $L, \mathfrak{D}(L)$, in $L^2(\Omega)$ can be taken to be those C^2 functions satisfying the boundary condi-