

A Priori Bounds for Solutions of Quasi-Linear Elliptic Differential Equations

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1. Introduction. Let Ω be a region in R^n , $n = 1, 2, \dots, 5$, $\partial\Omega$ denote its boundary. We consider quasi-linear elliptic differential equations of the form

$$(1) \quad L[u] \equiv - \sum_{|\alpha|, |\beta| \leq 1} D^\alpha (a_{\alpha\beta}(x) D^\beta u) = f(x, u), \quad x \in \Omega,$$

subject to the boundary conditions

$$(2) \quad u(x) = 0, \quad x \in \partial\Omega,$$

where we have freely used the standard multi-index notation, *cf.* [15]. For example, $x = (x_1, \dots, x_n) \in R^n$ and if $\alpha = (\alpha_1, \dots, \alpha_n)$ is any index whose components are non-negative integers, $|\alpha| \equiv \alpha_1 + \alpha_2 + \dots + \alpha_n$ and $D^\alpha \equiv D_1^{\alpha_1} \dots D_n^{\alpha_n} \equiv \partial^{\alpha_1 + \dots + \alpha_n} / \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$. A basic question, both in proving the existence of a solution of (1), (2) by means of the Schauder Fixed Point Theorem, *cf.* [1], [8], and in approximating that solution numerically, *cf.* [5], [6], [7], [11], [12], and [13] is whether or not we can obtain an *a priori* bound for classical solutions of (1), (2) in the uniform norm over Ω .

The special case of $n = 2$, $L \equiv -\Delta$ has been studied by many people. In [6] a uniform norm *a priori* bound was obtained for the case in which $\partial f / \partial u \leq \gamma < 1/\rho$, where $\rho \equiv \max_{x \in \Omega} |\psi(x)|$ and $\Delta\psi(x) = -1$, $x \in \Omega$, $\psi(x) = 0$, $x \in \partial\Omega$. In [9] such an *a priori* bound was obtained for the case in which

$$\liminf_{|u| \rightarrow \infty} f(x, u)/u \geq 0,$$

and in [11] such an *a priori* bound was obtained for the case in which there exists a positive constant u_0 such that $f(x, u)/u \leq -(1/a)^2$ for all $|u| \geq u_0$, where Ω is contained in the strip $|x_1| \leq a$. In this paper, we give new conditions on the problem (1), (2) which guarantee the existence of a uniform norm *a priori* bound.

2. Main results. We assume throughout this paper that the coefficients, $a_{\alpha\beta}(x)$, $|\alpha|, |\beta| \leq 1$, are real-valued, bounded, measurable functions in Ω and