

# Calculus of Set Valued Functions and Control

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**Introduction.** A beginning of what might be called a calculus of set valued functions can be found in [1]. Many references pertaining to this field are motivated by the mathematical theory of economics with problems of a similar nature appearing in the study of generalized differential equations having the form  $\dot{x}(t) \in R(t, x(t))$ , ( $\dot{x}(t) = d/dt x(t)$ ) where  $R$  is a set valued function. Such equations, commonly called contingent equations, arise naturally from control systems.

Following the notation of [1], if  $P$  is a function defined on a real interval  $[0, T]$  with values  $P(t)$  nonempty subsets of Euclidean  $n$  space,  $E^n$ , we define

$$\int_0^T P(\tau) d\tau = \left\{ \int_0^T p(\tau) d\tau : p \text{ is measurable, } p(\tau) \in P(\tau) \right\}.$$

We shall use measurable to mean Lebesgue measurable and prefix the word when it is used in a different sense.

It is always true that  $\int_0^T P(\tau) d\tau$  is convex. Furthermore, as shown in [1], if the graph of  $P$  is a Borel subset when considered as a subset of  $[0, T] \times E^n$ , then  $\int_0^T P(\tau) d\tau$  is nonempty. In this case  $P$  is termed Borel measurable. If  $P$  is Borel measurable with values subsets of a fixed compact subset of  $E^n$ ,  $\int_0^T P(\tau) d\tau = \int_0^T \text{co } P(\tau) d\tau$  where  $\text{co } P$  is the set valued function with values  $\text{co } P(t)$  the convex hull of  $P(t)$ . If, in addition, the values of  $P$  are closed subsets,  $\int_0^T P(\tau) d\tau$  is compact. These results will find constant use in this paper; they generalize previous work in [4] and [10].

If  $P$  is a Borel measurable set valued function with values nonempty, compact subsets of a given compact set in  $E^n$  and  $\mathfrak{A}(t) = \int_0^t P(\tau) d\tau$ , we shall call  $\mathfrak{A}$  the integral of  $P$ . In Section 1 we use a result of Pliś, [8], on a Lusin theorem for measurable set valued functions to show that  $\lim_{h \rightarrow 0} (1/h) \int_t^{t+h} P(\tau) d\tau = \text{co } P(t)$  for almost all  $t \in [0, T]$ . From this and Theorem 3 of [1], it can be shown that a necessary and sufficient condition that  $\int_0^t P^1(\tau) d\tau = \int_0^t P^2(\tau) d\tau$  for all  $t \in [0, T]$  is that  $\text{co } P^1(t) = \text{co } P^2(t)$  almost everywhere. Thus if  $\mathfrak{A}$  is the integral

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