

*Boundary Value Problems for Symmetric Positive Differential Operators of Odd Order**

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Communicated by RALPH PHILLIPS

Introduction. This paper treats a generalization to any odd order of the concept of first order symmetric positive systems of partial differential equations as introduced by K. O. Friedrichs [3]. In particular, we define concepts of weak and strong solutions of appropriate boundary value problems and show these concepts coincide (*i.e.*, “weak = strong”). This result is then used to establish existence and uniqueness of strong solutions of our symmetric positive boundary value problems.

We set up the problem in the following way. Let $k \geq 0$ be an integer. Let Ω be an open subset of E^p (real p -dimensional Euclidean space) with a smooth boundary β . $H_0^k(\Omega)$ is the usual Hilbert space of complex m -vector-valued functions obtained by completing $C_0^\infty(\Omega)$ under the norm induced by the inner product

$$(u, v)_k = \sum_{|\alpha| \leq k} (D^\alpha u, D^\alpha v) \quad (u, v \in C_0^\infty(\Omega)),$$

where $D^\alpha = D_1^{\alpha_1} \cdots D_p^{\alpha_p} = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_p)^{\alpha_p}$, $|\alpha| = \alpha_1 + \cdots + \alpha_p$ and (\cdot, \cdot) is the $\mathcal{L}_2(\Omega)$ inner product. Let $H_0^{-k}(\Omega)$ be the subset of distributions in Ω which have linear extensions from $C_0^\infty(\Omega)$ to $H_0^k(\Omega)$ which are continuous on $H_0^k(\Omega)$. With a suitable interpretation (see Sec. 1) $H_0^{-k}(\Omega)$ is the dual of $H_0^k(\Omega)$ with respect to an extension of the $\mathcal{L}_2(\Omega)$ inner product, and given the topology induced by this identification it is a Hilbert space. Differential operators of order at most $2k$ define bounded linear operators from $H_0^k(\Omega)$ into $H_0^{-k}(\Omega)$.

Now let

$$L = \sum_{|\alpha| \leq 2k+1} A_\alpha D^\alpha$$

be a differential expression of order $2k + 1$ where the A_α are smooth $m \times m$ matrix valued functions in $\Omega \cup \beta$. Let N be a function mapping $x \in \beta$ into $N(x)$,

* This research is supported in part by the Office of Naval Research Nonr 222(62) and the National Science Foundation Grant GP 7458.